

Option pricing in the large risk aversion, small transaction cost limit

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Abstract

We characterize the price of a European option on several assets for a very risk averse seller, in a market with small transaction costs as a solution of a nonlinear diffusion equation. This problem turns out to be one of asymptotic analysis of nonlinear parabolic PDE, and the interesting feature is the role of a nonlinear PDE eigenvalue problem. In particular, we generalize previous work of G. Barles and H. Soner who studied this problem for a European call option on a single asset where the associated eigenvalue problem involves an ODE with an explicit solution.

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1 Introduction

In their celebrated paper [4], F. Black and M. Scholes derived a formula for the fair price of a European call option on a single stock in an arbitrage free market. They also presented a “replication portfolio” that enabled the issuer of the option to hedge his position upon selling the option. The Black-Scholes model presented the first rational method for valuing options, and consequently, this model has been used in a large number of industrial applications.

Aside from the financial implications, interesting mathematics also came out of this work as the Black-Scholes formula is a solution of a certain *linear*, parabolic PDE

$$\psi_t + \frac{1}{2}\sigma^2 p^2 \psi_{pp} + rp\psi_p - r\psi = 0,$$

now known as the Black-Scholes equation. The purpose of this paper is to discuss an extension of the Black-Scholes model and further connections between *non-linear* PDE and option pricing.

The Black-Scholes model, being the first of its kind, has various shortcomings. One such shortcoming is the assumption that there are no costs for making transactions; in fact, in the Black-Scholes model, the issuer of an option is trying to hedge his position at each moment of time and thus transaction costs would be ruinous. This fact has been formalized and proved rigorously [21]. Another shortcoming of the model, is that it does not account for risk preferences of option issuers or purchasers; option prices are the same for buyers and sellers and each price is completely determined by known market parameters and the option’s payoff.

An alternative model, that addresses the aforementioned modeling issues, was presented by Davis, Panas, and Zariphopoulou [7]. This model (which we will call the DPZ model) uses the principle of certainty equivalent amount to define option prices and poses the option valuation problem as a problem of stochastic control theory. Within the DPZ model, G. Barles and H. Soner [2] discovered that in markets with small proportional transaction costs $\approx \sqrt{\epsilon}$, the asking price z^ϵ of a European option by a very risk averse $\approx \frac{1}{\epsilon}$ seller is approximately given by

$$z^\epsilon(t, p, y) \approx \psi(t, p) + \epsilon u \left(p \frac{\psi_p(t, p) - y}{\sqrt{\epsilon}} \right),$$

as ϵ tends to 0 [2]. Here ψ is a solution of a PDE resembling a non-linear version of the Black-Scholes equation

$$\psi_t + e^{-r(T-t)} \lambda(e^{r(T-t)} p^2 \psi_{pp}) + rp\psi_p - r\psi = 0,$$

and λ and the function u arising in the error term for z^ϵ together satisfy the ODE *eigenvalue problem*: for each $A \in \mathbb{R}$, find $\lambda(A)$ and $x \mapsto u = u(x; A)$ satisfying

$$\max \left\{ \lambda - \frac{\sigma^2}{2} (A + A^2 u'' + (x + Au')^2), |u'| - 1 \right\} = 0. \quad (1.1)$$

Establishing the convergence of z^ϵ to ψ , as ϵ tends to 0, is a problem of asymptotic analysis of parabolic PDE as z^ϵ is a solution of the PDE

$$\max \left\{ -z_t - \frac{1}{2} \sigma^2 p^2 \left(z_{pp} + \frac{1}{\epsilon} (z_p - y)^2 \right), |z_y| - \sqrt{\epsilon} p \right\} = 0.$$

Given the above convergence result, it is very natural to ask if similar phenomena occurs for European options on several assets. That is, in the DPZ model for option pricing on several assets, does the large risk aversion, small transaction option price exist? And if so, can it be characterized as a solution of a nonlinear Black-Scholes type of PDE? The purpose of this paper is to establish that this is indeed the case. The main novelty of this work is our treatment of the analog of eigenvalue problem described above. G. Barles and H. Soner observed that equation (1.1) has a near explicit solution, and this seems to be far from the case in the several asset setting; see Theorem 1.1 below.

1.1 Statement of results

We consider solutions $z^\epsilon = z^\epsilon(t, p, y)$ of the backwards parabolic equation

$$\max_{1 \leq i \leq n} \left\{ -z_t - \frac{1}{2} \text{tr} \left(d(p) \sigma \sigma^t d(p) \left(D_p^2 z + \frac{1}{\epsilon} (D_p z - y) \otimes (D_p z - y) \right) \right), |z_{y_i}| - \sqrt{\epsilon} p_i \right\} = 0,^1 \quad (1.2)$$

where $(t, p, y) \in (0, T) \times (0, \infty)^n \times \mathbb{R}^n$, that satisfy

$$z(T, p, y) = g(p). \quad (1.3)$$

Here

$$d(p) := \text{diag}(p_1, p_2, \dots, p_n) = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{pmatrix}$$

and it is assumed that ϵ and T are positive and that σ is a nonsingular $n \times n$ matrix. Our goal is to understand the behavior of solutions when ϵ tends to 0.

In analogy with the aforementioned work of G. Barles and H. Soner [2], we shall see that

$$z^\epsilon(t, p, y) \approx \psi(t, p) + \epsilon u \left(d(p) \frac{D\psi(t, p) - y}{\sqrt{\epsilon}} \right), \quad (1.4)$$

¹ For $a, b \in \mathbb{R}^n$, $a \otimes b$ is the $n \times n$ matrix with i, j th entry $a_i b_j$.

as ϵ tends to 0, where ψ is a solution of the *non-linear Black-Scholes equation*

$$\begin{cases} \psi_t + e^{-r(T-t)} \lambda \left(e^{r(T-t)} d(p) D^2 \psi d(p) \right) + r p \cdot D\psi - r\psi = 0, & (t, p) \in (0, T) \times (0, \infty)^n \\ \psi = g, & (t, p) \in \{T\} \times (0, \infty)^n \end{cases} \quad (1.5)$$

As in the single asset case, we will also see that the non-linearity λ and the function u together satisfy the following PDE eigenvalue problem: for each $A \in \mathcal{S}(n)$,² find $\lambda(A)$ and $x \mapsto u = u(x; A)$ satisfying

$$\max_{1 \leq i \leq n} \left\{ \lambda - \frac{1}{2} \text{tr} \sigma \sigma^t \left(A + AD^2 u A + (x + ADu) \otimes (x + ADu) \right), |u_{x_i}| - 1 \right\} = 0. \quad (1.6)$$

Our first theorem is

Theorem 1.1. *(Solution of the eigenvalue problem) For each $A \in \mathcal{S}(n)$, there is a unique $\lambda = \lambda(A)$ such that (1.6) has a viscosity solution $u \in C(\mathbb{R}^n)$ satisfying*

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} = 1. \quad (1.7)$$

Moreover, associated to $\lambda(A)$ is a convex solution u satisfying (1.7); when $\det A \neq 0$, $u \in C_{loc}^{1,\alpha}(\mathbb{R}^n)$ for each $0 < \alpha < 1$.

It follows from Theorem 1.1 that the eigenvalue problem associated to the PDE (1.6) has a well defined solution $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$. In order to properly interpret solutions of (1.5), we will need to know that λ is a nondecreasing function with respect to the partial ordering on $\mathcal{S}(n)$. In fact, we show more.

Theorem 1.2. *(Properties of λ) Let $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$ be as described in the statement of Theorem 1.1. Then*

- (i) λ is nondecreasing,
- (ii) λ is convex,
- (iii) for each $A \in \mathcal{S}(n)$ and each permutation matrix U ,

$$\lambda(UAU^t) = \lambda(A),$$

and (iv) $\lambda_- \leq \lambda \leq \lambda_+$, where

$$\begin{aligned} \lambda_-(A) = \sup \left\{ \inf_{x \in \mathbb{R}^n} \frac{1}{2} \text{tr} \sigma \sigma^t \left(A + AD^2 \phi(x) A + (x + AD\phi(x)) \otimes (x + AD\phi(x)) \right) : \right. \\ \left. \phi \in C^2(\mathbb{R}^n), \max_{1 \leq i \leq n} |\phi_{x_i}| \leq 1 \right\} \end{aligned}$$

² $\mathcal{S}(n)$ denotes the set of real symmetric, $n \times n$ matrices.

and

$$\lambda_+(A) = \inf \left\{ \sup_{\max_i |\psi_{x_i}(x)| < 1} \frac{1}{2} \text{tr} \sigma^t (A + AD^2 \psi(x) A + (x + AD \psi(x)) \otimes (x + AD \psi(x))) : \right. \\ \left. \psi \in C^2(\mathbb{R}^n), \liminf_{|x| \rightarrow \infty} \frac{\psi(x)}{\sum_{i=1}^n |x_i|} \geq 1 \right\}.$$

Furthermore, $\lambda_-(A) = \lambda(A) = \lambda_+(A)$, provided $\det A \neq 0$.

Employing the eigenvalue function λ and making some natural assumptions, we establish the following theorem, which is the main result of this paper.

Theorem 1.3. *(Convergence of solutions) Assume that $g \in C((0, \infty)^n)$. Then for each $\epsilon, T > 0$ there is a viscosity solution $z^\epsilon \in C((0, T] \times (0, \infty)^n \times \mathbb{R}^n)$ of (1.2) that satisfies (1.3). Further suppose that*

$$0 \leq g(p) \leq L \tag{1.8}$$

or

$$0 \leq g(p) \leq L \sum_{i=1}^n p_i \quad \text{and} \quad \lim_{|p| \rightarrow \infty} \frac{g(p)}{\sum_{i=1}^n p_i} = L, \tag{1.9}$$

for a given constant L . Then, as ϵ tends to 0, z^ϵ converges uniformly on compact subsets of $(0, T) \times (0, \infty)^n \times \mathbb{R}^n$ to a viscosity solution of equation (1.5).

In section 2, we study the eigenvalue problem in detail and prove Theorem 1.1. In section 3, we prove Theorem 1.2 which verifies some important properties of λ . Finally in section 4, we establish Theorem 1.3, which characterizes the large risk aversion, small transaction cost option price. Before undertaking this work, we present the mathematical model from which the equations derive and perform some formal computations that will guide our intuition for analyzing z^ϵ for ϵ small.

1.2 Model and formal asymptotics

The market model. Following the work of [7, 2, 9], we consider a Brownian motion based financial market consisting of n stocks and a money market account (a “bond”) with interest rate $r \geq 0$. The stocks are modeled as a stochastic process satisfying the SDE

$$dP^i(s) = \sum_{j=1}^n \sigma_{ij} P^i(s) dW^j(s), \quad s \geq 0, \quad i = 1, \dots, n$$

where $(W(t), t \geq 0)$ is a standard n -dimensional Brownian motion and σ is a non-singular $n \times n$ matrix. We assume each participant in the market assumes a trading strategy which is simply a way of purchasing and selling shares of stock and the money market account. Furthermore, in this model we assume that participants pay transaction costs that are proportional to the amount of the underlying stock; the proportionality constant we use is $\sqrt{\epsilon}$.

On a time interval $[t, T]$, a trading strategy will be modeled by a pair of vector processes $(L, M) = ((L^1, \dots, L^n), (M^1, \dots, M^n))$. Here $L^i(s)$ represents the cumulative purchases of the i th stock and $M^i(s)$ represents the cumulative sales of the i th stock at time $s \in [t, T]$; we assume L^i, M^i are non-decreasing processes, adapted to the filtration generated by W , that satisfy $L^i(t) = M^i(t) = 0$ for $i = 1, \dots, n$. Associated to a given trading strategy (L, M) is a process X , the amount of dollars held in the money market, and processes Y^i , the number of shares of the i th stock held, for $i = 1, \dots, n$. These processes are modeled by the SDE

$$\begin{cases} dX(s) = rX(s)ds + \sum_{i=1}^n (-(1 + \sqrt{\epsilon})P^i(s)dL^i(s) + (1 - \sqrt{\epsilon})P^i(s)dM^i(s)) \\ dY^i(s) = dL^i(s) - dM^i(s) \quad i = 1, \dots, n \end{cases} \quad t \leq s \leq T.$$

We assume that for a given amount of wealth $w \in \mathbb{R}$, a seller of a European option with maturity T and payoff $g(P(T)) \geq 0$ has the *utility*

$$U_\epsilon(w) = 1 - e^{-w/\epsilon}.$$

In particular, the seller has constant *risk aversion*

$$\frac{-U''_\epsilon(w)}{U'_\epsilon(w)} = \frac{1}{\epsilon}.$$

If the seller does not sell the option, his expected utility from final wealth is

$$v^{\epsilon, f}(t, x, y, p) = \sup_{L, M} \mathbb{E}U_\epsilon(X(T) + Y(T) \cdot P(T)).^3$$

If he does sell the option, he will have to payout $g(P(T))$ at time T , so his expected utility from final wealth is

$$v^\epsilon(t, x, y, p) = \sup_{L, M} \mathbb{E}U_\epsilon(X(T) + Y(T) \cdot P(T) - g(P(T))).$$

Note that since U_ϵ is monotone increasing, $v^\epsilon \leq v^{\epsilon, f}$. We define the seller's price Λ_ϵ as the amount which offsets this difference (and makes the seller "indifferent" to selling the option or not)

$$v^\epsilon(t, x + \Lambda_\epsilon, y, p) = v^{\epsilon, f}(t, x, y, p).$$

See [5] for more on this approach to option pricing and for more on the theory of indifference pricing.

As in the single asset case [7], we have the following proposition. Part (i) follows directly from Theorem 2 and Theorem 3 of [7]; part (ii) follows from basic calculus.

Proposition 1.4. (i) $v^\epsilon, v^{\epsilon, f}$ are the unique continuous viscosity solutions of the PDE

$$\max_{1 \leq i \leq n} \left\{ v_{y_i} - (1 + \sqrt{\epsilon})p_i v_x, -v_{y_i} + (1 - \sqrt{\epsilon})p_i v_x, v_t + \frac{1}{2}d(p)\sigma\sigma^t d(p) \cdot D_p^2 v + rp \cdot D_p v + rxv_x \right\} = 0, \quad (1.10)$$

³Here, and below, we are assuming that $X(t) = x, Y(t) = y$ and $P(t) = p$.

for $(t, y, p) \in (0, T) \times \mathbb{R}^n \times (0, \infty)^n$, that satisfy

$$v^\epsilon(T, x, y, p) = 1 - \exp(-(x + y \cdot p - g(p))/\epsilon) \quad \text{and} \quad v^{\epsilon, f}(T, x, y, p) = 1 - \exp(-(x + y \cdot p)/\epsilon).$$

(ii) Define $z^\epsilon, z^{\epsilon, f}$ implicitly via

$$\begin{cases} v^\epsilon = U_\epsilon(x + y \cdot p - z^\epsilon) = 1 - \exp(-(x + y \cdot p - z^\epsilon)/\epsilon) \\ v^{\epsilon, f} = U_\epsilon(x + y \cdot p - z^{\epsilon, f}) = 1 - \exp(-(x + y \cdot p - z^{\epsilon, f})/\epsilon) \end{cases}.$$

Then $z^\epsilon, z^{\epsilon, f}$ are viscosity solutions of (1.2) satisfying the terminal conditions

$$z^\epsilon(T, p, y) = g(p) \quad \text{and} \quad z^{\epsilon, f}(T, p, y) = 0.$$

Important reductions. (a) To simplify the presentation, we set $r = 0$. However, this is done without any loss of generality as the function

$$\tilde{v}(t, x, y, p) := v(t, e^{-r(T-t)}x, y, e^{-r(T-t)}p) \quad (1.11)$$

satisfies the PDE (1.10) with $r = 0$, provided of course that v is a solution of (1.10). Moreover, $\tilde{v}(T, x, y, p) = v(T, x, y, p)$.

(b) The main virtue of working with the exponential utility function is that the value functions typically depend on the x variable in a simple way. Notice that (upon setting $r = 0$)

$$X^{t,x}(s) = x + \int_t^s \{-(1 + \sqrt{\epsilon})P(s) \cdot dL(s) + (1 - \sqrt{\epsilon})P(s) \cdot dM(s)\}, \quad t \leq s \leq T$$

and so $v = v^\epsilon, v^{\epsilon, f}$ satisfy

$$v(t, x, y, p) = 1 + e^{-x/\epsilon}(v(t, 0, y, p) - 1). \quad (1.12)$$

This is convenient as it reduces the variable dependence of solutions of (1.10). Moreover, using (1.12), it is straightforward to check that

$$z^\epsilon, z^{\epsilon, f} \text{ are independent of } x.$$

The large risk aversion, small transaction cost limit. Directly from the definition of Λ_ϵ and the definition of $z^\epsilon, z^{\epsilon, f}$, we see

$$\Lambda_\epsilon = z^\epsilon - z^{\epsilon, f}.$$

Consequently, in order to understand the limiting option price it suffices to study $\lim_{\epsilon \rightarrow 0^+} z^\epsilon$ and $\lim_{\epsilon \rightarrow 0^+} z^{\epsilon, f}$. Therefore, the problem of characterizing the limiting option price is reduced to that of asymptotic analysis of solutions nonlinear parabolic PDE.

Below, we give a step-by-step formal derivation of how we arrived at the PDE [equation (1.5)] for the limit ψ and the PDE [equation (1.6)] arising in the eigenvalue problem. These

heuristic calculations are arguably the most important part of our work since the techniques we later use are founded on these results. These computations are based largely on section 3.2 of [2].

Step 1. $\max_{1 \leq i \leq n} \{|z_{y_i}^\epsilon| - \sqrt{\epsilon} p_i\} \leq 0$, so we expect $\lim_{\epsilon \rightarrow 0^+} z^\epsilon$ to be independent of y . This observation leads to the choice of *ansatz*

$$z^\epsilon(t, p, y) \approx \psi(t, p) + \epsilon u(x^\epsilon(t, p, y)),$$

for ϵ small. Here ψ , u and x^ϵ are yet to be determined. Using this ansatz, we formally compute

$$\begin{aligned} z_t^\epsilon &\approx \psi_t + \epsilon Du(x^\epsilon) \cdot x_t^\epsilon \\ D_y z^\epsilon &\approx \epsilon (D_y x^\epsilon)^t Du(x^\epsilon) \\ D_p z^\epsilon &\approx D\psi + \epsilon (D_p x^\epsilon)^t Du(x^\epsilon) \\ D_p^2 z^\epsilon &\approx D^2 \psi + \epsilon ((D_p x^\epsilon)^t D^2 u(x^\epsilon) D_p x^\epsilon + D_p^2 x^\epsilon \cdot Du(x^\epsilon)) \end{aligned}$$

where $(D_p^2 x^\epsilon \cdot Du(x^\epsilon))_{ij} := x_{p_i p_j}^\epsilon \cdot Du(x^\epsilon)$, $i, j = 1, \dots, n$.

Step 2. We also observe that since

$$\epsilon |x_{y_i}^\epsilon \cdot Du(x^\epsilon)| \approx |z_{y_i}^\epsilon| \leq \sqrt{\epsilon} p_i,$$

x^ϵ (and its derivatives) should probably scale at worst like $1/\sqrt{\epsilon}$. With this assumption, we calculate

$$\begin{aligned} I^\epsilon &:= -z_t - \frac{1}{2} \text{tr} \left(d(p) \sigma \sigma^t d(p) \left(D_p^2 z + \frac{1}{\epsilon} (D_p z - y) \otimes (D_p z - y) \right) \right) \\ &\approx -\psi_t - \frac{1}{2} \text{tr} \sigma \sigma^t \left(d(p) D^2 \psi d(p) + (\sqrt{\epsilon} D_p x^\epsilon d(p))^t D^2 u(x^\epsilon) (\sqrt{\epsilon} D_p x^\epsilon d(p)) + \right. \\ &\quad \left. \left(d(p) \frac{D\psi - y}{\sqrt{\epsilon}} + (\sqrt{\epsilon} D_p x^\epsilon d(p))^t Du(x^\epsilon) \right) \otimes \left(d(p) \frac{D\psi - y}{\sqrt{\epsilon}} + (\sqrt{\epsilon} D_p x^\epsilon d(p))^t Du(x^\epsilon) \right) \right). \end{aligned}$$

Step 3. Notice that

$$\sqrt{\epsilon} D_p \left(d(p) \frac{D\psi - y}{\sqrt{\epsilon}} \right) d(p) = d(p) D^2 \psi d(p) + \sqrt{\epsilon} \begin{pmatrix} p_1 \frac{\psi_{p_1} - y_1}{\sqrt{\epsilon}} & & & \\ & p_2 \frac{\psi_{p_2} - y_2}{\sqrt{\epsilon}} & & \\ & & \ddots & \\ & & & p_n \frac{\psi_{p_n} - y_n}{\sqrt{\epsilon}} \end{pmatrix}.$$

This basic observation and the above computations lead us to choose the new “variable”

$$x^\epsilon := d(p) \frac{D\psi - y}{\sqrt{\epsilon}}$$

and the new “parameter”

$$A := d(p)D^2\psi d(p).$$

We further *postulate* that there is a function λ such that

$$\psi_t + \lambda(A) = 0.$$

Step 4. With the above choices and postulate,

$$I^\epsilon \approx \lambda(A) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u(x^\epsilon) A + (x^\epsilon + ADu(x^\epsilon)) \otimes (x^\epsilon + ADu(x^\epsilon)))$$

and also for $i = 1, \dots, n$

$$|u_{x_i}(x^\epsilon)| \lesssim 1.$$

Since

$$\max_{1 \leq i \leq n} \{I^\epsilon, |z_{y_i}| - \sqrt{\epsilon} p_i\} = 0,$$

we will require that u and $\lambda(A)$ satisfy

$$\max_{1 \leq i \leq n} \left\{ \lambda - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u A + (x + ADu) \otimes (x + ADu)), |u_{x_i}| - 1 \right\} = 0$$

for $x \in \mathbb{R}^n$. In view of estimates we will later derive on z^ϵ (see inequality (4.10)), we additionally require

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} = 1.$$

In summary, we have the following **nonlinear eigenvalue problem**:

For $A \in \mathcal{S}(n)$, find $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} \max_{1 \leq i \leq n} \left\{ \lambda - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u A + (x + ADu) \otimes (x + ADu)), |u_{x_i}| - 1 \right\} = 0, & x \in \mathbb{R}^n \\ \lim_{|x| \rightarrow \infty} u(x) / \sum_{i=1}^n |x_i| = 1 \end{cases}.$$

If we can solve the above eigenvalue problem *uniquely for a nondecreasing function* λ , we have the solution of the PDE (1.5) as a candidate for the limit of z^ϵ as $\epsilon \rightarrow 0^+$. We remark that the procedure described above is philosophically similar to the formal asymptotics of periodic homogenization. In analogy with that framework, λ plays the role of the effective Hamiltonian, and the eigenvalue problem plays the role of the cell problem [20, 12]. Finally, we note that the same heuristic argument shows that $z^{\epsilon, f}$ satisfies the PDE (1.5) except with $\psi|_{t=T} \equiv 0$ terminal condition. We formally conclude that $\lim_{\epsilon \rightarrow 0^+} z^{\epsilon, f} = 0$ and in particular,

$$\lim_{\epsilon \rightarrow 0^+} \Lambda_\epsilon = \lim_{\epsilon \rightarrow 0^+} z^\epsilon.$$

Remark 1.5. It is possible to use the relationship (1.11) to deduce that if ψ satisfies (1.5) with $r = 0$, then

$$(t, p) \mapsto e^{-r(T-t)} \psi(t, e^{r(T-t)} p)$$

satisfies (1.5) (for nonzero r). Thus, without loss of generality, we will set $r = 0$ to simplify computations.

Modeling Remarks. (a) While Theorem 1.3 does not cover every possible payoff function g for a European option on several assets, it covers many that arise in practice. Some prototypical examples are:

Example 1.6. (Basket call option) $p \mapsto (\sum_{i=1}^n p_i - K)^+$

Example 1.7. (Basket of put options) $p \mapsto \sum_{i=1}^n (K - p_i)^+$

Example 1.8. (“Mixed” call option) $p \mapsto (p_1 - K)^+ + (p_1 + 2p_2 - K)^+ \quad (n = 2)$

Example 1.9. (Forward contract based on the geometric mean) $p \mapsto K - (\prod_{i=1}^n p_i)^{1/n}$.

(b) In view of the asymptotic expansion (1.4) and the growth condition (1.7), we have

$$\Lambda^\epsilon(t, p, y) \approx \psi(t, p) + \sum_{i=1}^n \sqrt{\epsilon} p_i |\psi_{p_i}(t, p) - y_i|$$

as $\epsilon \rightarrow 0^+$. Thus for small ϵ , Λ_ϵ is a sum of the limiting option price plus a term that naturally resembles a transaction cost.

2 A nonlinear eigenvalue problem

In this section, we prove the first part of Theorem 1.1 which is the statement that the eigenvalue problem is well posed. Our methods are largely based on the approach given in our previous article [18], however we consider this work a considerable extension. First, we give a definition that will allow for clear statements to follow. This definition involves viscosity solutions of nonlinear elliptic PDE and we refer readers to the standard sources for background material on this concept [6, 15, 1]. We shall also employ the notation of [6].

Definition 2.1. $u \in USC(\mathbb{R}^n)$ is a *viscosity subsolution* of (1.6) with *eigenvalue* $\lambda \in \mathbb{R}$ if for each $x_0 \in \mathbb{R}^n$,

$$\max \left\{ \lambda - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 \varphi(x_0) A + (x_0 + AD \varphi(x_0)) \otimes (x_0 + AD \varphi(x_0))), |D \varphi(x_0)| - 1 \right\} \leq 0,$$

whenever $u - \varphi$ has a local maximum at x_0 and $\varphi \in C^2(\mathbb{R}^n)$. $v \in LSC(\mathbb{R}^n)$ is a *viscosity supersolution* of (1.6) with *eigenvalue* $\mu \in \mathbb{R}$ if for each $y_0 \in \mathbb{R}^n$,

$$\max \left\{ \lambda - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 \psi(y_0) A + (y_0 + AD \psi(y_0)) \otimes (y_0 + AD \psi(y_0))), |D \psi(y_0)| - 1 \right\} \geq 0,$$

whenever $v - \psi$ has a local minimum at y_0 and $\psi \in C^2(\mathbb{R}^n)$. $u \in C(\mathbb{R}^n)$ is a *viscosity solution* of (1.6) with *eigenvalue* $\lambda \in \mathbb{R}$ if its both a viscosity sub- and supersolution of (1.6) with eigenvalue λ .

2.1 Comparison of eigenvalues

We start our treatment of the eigenvalue problem by establishing a fundamental comparison principle that will allow us to compare eigenvalues associated with sub- and supersolutions of (1.6).

Proposition 2.2. *Suppose u is a subsolution of (1.6) with eigenvalue λ and that v is a supersolution of (1.6) with eigenvalue μ . If in addition*

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} \leq 1 \leq \liminf_{|x| \rightarrow \infty} \frac{v(x)}{\sum_{i=1}^n |x_i|}, \quad (2.1)$$

then $\lambda \leq \mu$.

We will first give a heuristic proof, which will naturally motivate the rigorous argument to follow. This will be a common theme in our work.

Formal proof: Assume that $u, v \in C^2(\mathbb{R}^n)$. Fix $0 < \tau < 1$ and set

$$w^\tau(x) = \tau u(x) - v(x), \quad x \in \mathbb{R}^n.$$

By (2.1), we have that $\lim_{|x| \rightarrow \infty} w^\tau(x) = -\infty$, so there is $x_\tau \in \mathbb{R}^n$ such that

$$w^\tau(x_\tau) = \sup_{x \in \mathbb{R}^n} w^\tau(x).$$

Basic calculus gives

$$\begin{cases} 0 = Dw^\tau(x_\tau) = \tau Du(x_\tau) - Dv(x_\tau) \\ 0 \geq D^2w^\tau(x_\tau) = \tau D^2u(x_\tau) - D^2v(x_\tau) \end{cases}.$$

Note in particular that

$$|v_{x_i}(x_\tau)| = \tau |u_{x_i}(x_\tau)| \leq \tau < 1, \quad i = 1, \dots, n$$

and since v is a supersolution of (1.6) with eigenvalue μ ,

$$\mu - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2v(x_\tau)A + (x_\tau + ADv(x_\tau)) \otimes (x_\tau + ADv(x_\tau))) \geq 0.$$

As u is a subsolution of (1.6) with eigenvalue λ ,

$$\begin{aligned}
\tau\lambda - \mu &\leq \frac{1}{2}\text{tr}\sigma\sigma^t [(\tau-1)A + A(\tau D^2u(x_\tau) - D^2v(x_\tau))A \\
&\quad + \tau(x_\tau + ADu(x_\tau)) \otimes (x_\tau + ADu(x_\tau)) - (x_\tau + ADv(x_\tau)) \otimes (x_\tau + ADv(x_\tau))] \\
&\leq \frac{1}{2}\text{tr}\sigma\sigma^t [(\tau-1)A + \tau(x_\tau + ADu(x_\tau)) \otimes (x_\tau + ADu(x_\tau)) \\
&\quad - (x_\tau + \tau ADu(x_\tau)) \otimes (x_\tau + \tau ADu(x_\tau))] \\
&= \frac{1}{2}\text{tr}\sigma\sigma^t [(\tau-1)(A + x_\tau \otimes x_\tau) + \tau(1-\tau)ADu(x_\tau) \otimes ADu(x_\tau)] \\
&\leq \frac{1}{2}(\tau-1)\text{tr}\sigma\sigma^t A + \frac{1}{2}\tau(1-\tau)|\sigma^t ADu(x_\tau)|^2 \\
&\leq \frac{1}{2}(\tau-1)\text{tr}A + \frac{n}{2}\tau(1-\tau)|\sigma^t A|^2.
\end{aligned}$$

We conclude by letting $\tau \rightarrow 1^-$.

□

Proof. We now employ a “doubling the variables” argument to make this above heuristic proof rigorous. The main difference with standard arguments is that we are working on the entire space \mathbb{R}^n . In the proof below, we will quote a few basic facts about semicontinuous functions established in [6].

1. Fix $0 < \tau < 1$ and set

$$w^\tau(x, y) = \tau u(x) - v(y), \quad x, y \in \mathbb{R}^n.$$

For $\delta > 0$, we also set

$$\varphi_\delta(x, y) = \frac{1}{2\delta}|x - y|^2, \quad x, y \in \mathbb{R}^n.$$

The inequality

$$\begin{aligned}
w^\tau(x, y) - \varphi_\delta(x, y) &= \tau(u(x) - u(y)) - \frac{1}{2\delta}|x - y|^2 + \tau u(y) - v(y) \\
&\leq \left(\sqrt{n}|x - y| - \frac{1}{2\delta}|x - y|^2 \right) + \tau u(y) - v(y)
\end{aligned}$$

implies

$$\lim_{|(x, y)| \rightarrow \infty} \{w^\tau(x, y) - \varphi_\delta(x, y)\} = -\infty.$$

Therefore, $w^\tau - \varphi_\delta$ achieves a global maximum at a point $(x_\delta, y_\delta) \in \mathbb{R}^n \times \mathbb{R}^n$.

2. According to the Theorem of Sums (Theorem 3.2 in [6]), for each $\rho > 0$, there are $X, Y \in \mathcal{S}(n)$ such that

$$\left(\frac{x_\delta - y_\delta}{\delta}, X \right) = (D_x \varphi_\delta(x_\delta, y_\delta), X) \in \overline{J}^{2,+}(\tau u)(x_\delta),$$

$$\left(\frac{x_\delta - y_\delta}{\delta}, Y\right) = (-D_y \varphi_\delta(x_\delta, y_\delta), Y) \in \bar{J}^{2,-} v(y_\delta), \quad (2.2)$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \rho A^2. \quad (2.3)$$

Here

$$A = D^2 \varphi_\delta(x_\delta, y_\delta) = \frac{1}{\delta} \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}.$$

Note that (2.3) implies that $X \leq Y$.

3. Set

$$p_\delta = \frac{1}{\tau} \frac{x_\delta - y_\delta}{\delta}$$

and note that $p_\delta \in \bar{J}^{1,+} u(x_\delta)$. Also note that as $\max_{1 \leq i \leq n} |u_{x_i}| \leq 1$ (in the sense of viscosity solutions),

$$\max_{1 \leq i \leq n} |\tau p_\delta \cdot e_i| \leq \tau < 1.^4$$

Since v is a viscosity super-solution of (1.6) with eigenvalue μ , we have

$$\mu - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AY A + (y_\delta + \tau A p_\delta) \otimes (y_\delta + \tau A p_\delta)) \geq 0$$

by (2.2). As u is a viscosity sub-solution of (1.6) with eigenvalue λ ,

$$\lambda - \frac{1}{2} \text{tr} \sigma \sigma^t (A + A(X/\tau)A + (x_\delta + A p_\delta) \otimes (x_\delta + A p_\delta)) \leq 0.$$

Therefore,

$$\begin{aligned} \tau \lambda - \mu &\leq \frac{1}{2}(\tau - 1) \text{tr} \sigma \sigma^t A + \frac{1}{2} \text{tr} \sigma \sigma^t [A(X - Y)A] + \frac{1}{2} \tau |\sigma^t(x_\delta + A p_\delta)|^2 - \frac{1}{2} |\sigma^t(y_\delta + \tau A p_\delta)|^2 \\ &\leq \frac{1}{2}(\tau - 1) \text{tr} \sigma \sigma^t A + \frac{1}{2} \tau |\sigma^t(x_\delta + A p_\delta)|^2 - \frac{1}{2} |\sigma^t(y_\delta + \tau A p_\delta)|^2. \end{aligned} \quad (2.4)$$

4. We now claim that $x_\delta \in \mathbb{R}^n$ is bounded for all small enough $\delta > 0$. If not, then there is a subsequence of $\delta \rightarrow 0$ such that $(w^\tau - \varphi_\delta)(x_\delta, y_\delta)$ tends to $-\infty$. Indeed

$$\begin{aligned} (w^\tau - \varphi_\delta)(x_\delta, y_\delta) &= (\tau u(x_\delta) - v(x_\delta)) + v(x_\delta) - v(y_\delta) - \frac{|x_\delta - y_\delta|^2}{2\delta} \\ &\leq (\tau u(x_\delta) - v(x_\delta)) + \sqrt{n} |x_\delta - y_\delta| - \frac{|x_\delta - y_\delta|^2}{2\delta} \\ &\leq (\tau u(x_\delta) - v(x_\delta)) + \frac{n\delta}{2} \end{aligned} \quad (2.5)$$

which tends to $-\infty$ as $\delta \rightarrow 0$ provided $\lim_{\delta \rightarrow 0^+} |x_\delta| = +\infty$. This would be the case for some sequence of $\delta \rightarrow 0$ if x_δ is unbounded.

⁴Here $\{e_1, \dots, e_n\}$ denotes the standard basis in \mathbb{R}^n .

However,

$$\begin{aligned}
(w^\tau - \varphi_\delta)(x_\delta, y_\delta) &= \max_{x, y \in \mathbb{R}^n} \left\{ \tau u(x) - v(y) - \frac{|x - y|^2}{2\delta} \right\} \\
&\geq \tau u(0) - v(0) \\
&> -\infty
\end{aligned} \tag{2.6}$$

and thus x_δ lies in a bounded subset of \mathbb{R}^n . Since

$$\lim_{\delta \rightarrow 0^+} \frac{|x_\delta - y_\delta|^2}{2\delta} \rightarrow 0$$

(by Lemma 3.1 in [6]) and y_δ is bounded, the sequence $((x_\delta, y_\delta))_{\delta > 0}$ has a cluster point (x_τ, x_τ) for a sequence of $\delta \rightarrow 0$. Note also that p_δ is a bounded sequence so we can also assume that

$$p_\delta \rightarrow p$$

as $\delta \rightarrow 0$, for some $\max_{1 \leq i \leq n} |p_i| \leq 1$.

Passing to this limit in (2.4) gives

$$\begin{aligned}
\tau\lambda - \mu &\leq \frac{1}{2}(\tau - 1)\text{tr}\sigma\sigma^t A + \frac{1}{2}\tau|\sigma^t(x_\tau + Ap)|^2 - \frac{1}{2}|\sigma^t(x_\tau + \tau Ap)|^2 \\
&\leq \frac{1}{2}(\tau - 1)\text{tr}\sigma\sigma^t A + \frac{1}{2}(\tau - 1)|\sigma^t x_\tau|^2 + \frac{1}{2}\tau(1 - \tau)|\sigma^t Ap|^2 \\
&\leq \frac{1}{2}(\tau - 1)\text{tr}\sigma\sigma^t A + \frac{n}{2}\tau(1 - \tau)|\sigma^t A|^2
\end{aligned}$$

We conclude by letting $\tau \rightarrow 1^-$. □

The following corollary is immediate.

Corollary 2.3. *For each $A \in S(n)$, there can be at most one λ such that (1.6) has a solution u with eigenvalue λ satisfying the growth condition (1.7).*

Now that we know that there can be at most one solution of the eigenvalue problem we are left to answer the question of whether or not a single solution exists. We shall see that this is in fact the case. To approximate the values of a potential eigenvalue, we study the PDE

$$\max_{1 \leq i \leq n} \left\{ \delta u - \frac{1}{2}\text{tr}\sigma\sigma^t (A + AD^2 u A + (x + ADu) \otimes (x + ADu)), |u_{x_i}| - 1 \right\} = 0, \quad x \in \mathbb{R}^n \tag{2.7}$$

for $\delta > 0$ and small, and seek solutions that satisfy growth condition (1.7)

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} = 1.$$

The goal is to show that the above PDE has a unique solution u_δ and that there is a sequence of $\delta \rightarrow 0^+$ such that $\delta u_\delta(0) \rightarrow \lambda(A)$. Moreover, we hope that $u_\delta - u_\delta(0)$ converges to a solution u of (1.6). First, we address the question of uniqueness of solutions of (2.7). As this can be handled similar to the comparison principle for eigenvalues, we omit the proof.

Proposition 2.4. Suppose u is a subsolution of (2.7) and that v is a supersolution of (2.7). If in addition

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} \leq 1 \leq \liminf_{|x| \rightarrow \infty} \frac{v(x)}{\sum_{i=1}^n |x_i|},$$

then $u \leq v$.

Corollary 2.5. For each $A \in S(n)$, there can be at most one solution of (2.7) satisfying (1.7).

To establish existence, we need sub- and supersolutions with the appropriate growth as $|x| \rightarrow \infty$.

Lemma 2.6. Fix $0 < \delta < 1$ and $A \in \mathcal{S}(n)$.

(i) There is a constant $K = K(A) > 0$ such that

$$\underline{u}(x) = \left(\sum_{i=1}^n |x_i| - K \right)^+ + \frac{\text{tr} \sigma^t \sigma A}{2\delta}, \quad x \in \mathbb{R}^n \quad (2.8)$$

is a viscosity subsolution of (2.7) satisfying the growth condition (1.7).

(ii) There is a constant $K = K(A) > 0$ such that

$$\overline{u}(x) = \frac{K}{\delta} + \sum_{i=1}^n \begin{cases} \frac{1}{2} x_i^2, & |x_i| \leq 1 \\ |x_i| - \frac{1}{2}, & |x_i| \geq 1 \end{cases}, \quad x \in \mathbb{R}^n \quad (2.9)$$

is a viscosity supersolution of (2.7) satisfying the growth condition (1.7).

Proof. (i) Choose $K > 0$ such that

$$\left(\sum_{i=1}^n |x_i| - K \right)^+ \leq \frac{1}{2} \text{tr} \sigma \sigma^t A + \frac{1}{2} (|\sigma^t x| - \sqrt{n} |\sigma^t A|)^2, \quad x \in \mathbb{R}^n.$$

As \underline{u} is convex and as $\max_{1 \leq i \leq n} |\underline{u}_{x_i}| = 1$, if $(p, X) \in J^{2,+} \underline{u}(x_0)$ then

$$\max_{1 \leq i \leq n} |p_i| \leq 1 \quad \text{and} \quad X \geq 0.$$

Hence,

$$\begin{aligned} \delta \underline{u}(x_0) - \frac{1}{2} \text{tr} \sigma^t \sigma (A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)) &\leq \left(\sum_{i=1}^n |x_0 \cdot e_i| - K \right)^+ - \frac{1}{2} \text{tr} \sigma \sigma^t A \\ &\quad - \frac{1}{2} |\sigma^t (x_0 + Ap)|^2 \\ &\leq \left(\sum_{i=1}^n |x_0 \cdot e_i| - K \right)^+ - \frac{1}{2} \text{tr} \sigma \sigma^t A \\ &\quad - \frac{1}{2} (|\sigma^t x_0| - \sqrt{n} |\sigma^t A|)^2 \\ &\leq 0. \end{aligned}$$

Thus \underline{u} is a viscosity subsolution.

(ii) Choose

$$K := \max \left\{ \frac{1}{2} \text{tr} \sigma \sigma^t (A + A^2 + (I_n + A)x \otimes (I_n + A)x) : \max_{1 \leq i \leq n} |x_i| \leq 1 \right\}$$

and assume that $(p, X) \in J^{2,-} \underline{u}(x_0)$. If $|x_0 \cdot e_i| < 1$ for all $i = 1, \dots, n$ \bar{u} is smooth in a neighborhood of x_0 and

$$\begin{cases} \bar{u}(x_0) = \frac{K}{\delta} + \frac{|x_0|^2}{2} \\ D\bar{u}(x_0) = x_0 = p \\ D^2\bar{u}(x_0) = I_n = X \end{cases}.$$

Therefore,

$$\delta \bar{u}(x_0) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)) \geq K - \frac{1}{2} \text{tr} \sigma \sigma^t [A + A^2 + (I_n + A)x_0 \otimes (I_n + A)x_0] \geq 0,$$

which implies

$$\max_{1 \leq i \leq n} \left\{ \delta \bar{u}(x_0) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)), |p_i| - 1 \right\} \geq 0. \quad (2.10)$$

Now suppose $|x_0 \cdot e_i| \geq 1$ for some $i \in \{1, \dots, n\}$. $\bar{u} \in C^1(\mathbb{R}^n)$, so $p_i = \bar{u}_{x_i}(x_0) = x_0 \cdot e_i / |x_0 \cdot e_i|$ and in particular $|p_i| = 1$. Thus (2.10) still holds, and consequently, \bar{u} is a viscosity supersolution. \square

As the existence of a unique viscosity solution now follows directly from applying Perron's method of viscosity sub- and supersolutions (see section 4 of [6], for instance), we omit the proof.

Theorem 2.7. *Fix $0 < \delta < 1$ and $A \in \mathcal{S}(n)$. There exists a unique viscosity solution $u = u_\delta$ of the PDE (2.7) satisfying the growth condition (1.7).*

2.2 Basic estimates

With the existence of a unique solution of (2.7), our goal is establish some estimates on u_δ that will help us pass to the limit as $\delta \rightarrow 0$. A fundamental property of u_δ that we deduce below is that it is convex. Other important estimates of u_δ will be derived directly from this. The method of proof is virtually the same as in [18] (Lemma 3.7) and originates from the work of [19].

Proposition 2.8. *u_δ is convex.*

Proof. 1. We first assume $u \in C^2(\mathbb{R}^n)$ and for ease of notation, we write u for u_δ . Fix $0 < \tau < 1$ and set

$$C^\tau(x, y) = \tau u \left(\frac{x + y}{2} \right) - \frac{u(x) + u(y)}{2}, \quad x, y \in \mathbb{R}^n.$$

We aim to bound \mathcal{C}^τ from above and later send $\tau \rightarrow 1^-$.

2. As u grows like $\sum_{i=1}^n |x_i|$, as $|x| \rightarrow \infty$, it is straightforward to check that there is (x_τ, y_τ) maximizing \mathcal{C}^τ . At this point,

$$0 = D_x \mathcal{C}^\tau(x_\tau, y_\tau) = \frac{\tau}{2} Du\left(\frac{x_\tau + y_\tau}{2}\right) - \frac{1}{2} Du(x_\tau)$$

and

$$0 = D_y \mathcal{C}^\tau(x_\tau, y_\tau) = \frac{\tau}{2} Du\left(\frac{x_\tau + y_\tau}{2}\right) - \frac{1}{2} Du(y_\tau).$$

Thus,

$$\tau Du\left(\frac{x_\tau + y_\tau}{2}\right) = Du(x_\tau) = Du(y_\tau).$$

Also observe that $v \mapsto \mathcal{C}^\tau(x_\tau + v, y_\tau + v)$ has a maximum at $v = 0$ which implies

$$0 \geq \tau D^2 u\left(\frac{x_\tau + y_\tau}{2}\right) - \frac{D^2 u(x_\tau) + D^2 u(y_\tau)}{2}.$$

Since,

$$|u_{x_i}(x_\tau)| = |u_{x_i}(y_\tau)| = \tau \left| u_{x_i}\left(\frac{x_\tau + y_\tau}{2}\right) \right| \leq \tau < 1$$

for $i = 1, \dots, n$, we have

$$\delta u(z) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u(z)A + (z + ADu(z)) \otimes (z + ADu(z))) = 0, \quad z = x_\tau, y_\tau.$$

Set $z_\tau = (x_\tau + y_\tau)/2$, $p_\tau = Du(z_\tau)$, and notice

$$\begin{aligned} \delta \mathcal{C}^\tau(x, y) &\leq \delta \mathcal{C}^\tau(x_\tau, y_\tau) \\ &= \tau \delta u\left(\frac{x_\tau + y_\tau}{2}\right) - \frac{\delta u(x_\tau) + \delta u(y_\tau)}{2} \\ &\leq \frac{\tau}{2} \text{tr} \sigma \sigma^t (A + AD^2 u(z_\tau)A + (z_\tau + ADu(z_\tau)) \otimes (z_\tau + ADu(z_\tau))) \\ &\quad - \frac{1}{4} \text{tr} \sigma \sigma^t (A + AD^2 u(x_\tau)A + (x_\tau + ADu(x_\tau)) \otimes (x_\tau + ADu(x_\tau))) \\ &\quad - \frac{1}{4} \text{tr} \sigma \sigma^t (A + AD^2 u(y_\tau)A + (y_\tau + ADu(y_\tau)) \otimes (y_\tau + ADu(y_\tau))) \\ &= \frac{(\tau - 1)}{2} \text{tr} \sigma \sigma^t A + \frac{1}{2} \text{tr} \sigma \sigma^t \left[A \left(\tau D^2 u(z_\tau) - \frac{D^2 u(x_\tau) + D^2 u(y_\tau)}{2} \right) A \right] \\ &\quad + \frac{\tau}{2} |\sigma^t(z_\tau + ADu(z_\tau))|^2 - \frac{1}{4} |\sigma^t(x_\tau + ADu(x_\tau))|^2 - \frac{1}{4} |\sigma^t(y_\tau + ADu(y_\tau))|^2 \\ &\leq \frac{(\tau - 1)}{2} \text{tr} \sigma \sigma^t A + \frac{\tau}{2} |\sigma^t(z_\tau + Ap_\tau)|^2 - \frac{1}{4} |\sigma^t(x_\tau + \tau Ap_\tau)|^2 - \frac{1}{4} |\sigma^t(y_\tau + \tau Ap_\tau)|^2 \\ &= \frac{(\tau - 1)}{2} \text{tr} \sigma \sigma^t A + \frac{(\tau - 1)}{4} (|\sigma^t x_\tau|^2 + |\sigma^t y_\tau|^2) + \frac{1}{2} \tau (1 - \tau) |\sigma^t Ap_\tau|^2 \\ &\leq \frac{(\tau - 1)}{2} \text{tr} \sigma \sigma^t A + \frac{n}{2} \tau (1 - \tau) |\sigma^t A|^2, \end{aligned}$$

for each $x, y \in \mathbb{R}^n$. Sending $\tau \rightarrow 1^-$, we conclude that

$$u\left(\frac{x+y}{2}\right) - \frac{u(x) + u(y)}{2} \leq 0, \quad x, y \in \mathbb{R}^n.$$

3. To make this argument rigorous, we fix $0 < \tau < 1$ and now set

$$w^\tau(x, y, z) = \tau u(z) - \frac{u(x) + u(y)}{2}, \quad x, y, z \in \mathbb{R}^n;$$

and for $\eta > 0$, set

$$\varphi_\eta(x, y, z) = \frac{1}{2\eta} \left| z - \frac{x+y}{2} \right|^2, \quad x, y, z \in \mathbb{R}^n.$$

Notice that

$$\begin{aligned} (w^\tau - \varphi_\eta)(x, y, z) &= \tau \left\{ u(z) - u\left(\frac{x+y}{2}\right) \right\} - \frac{1}{2\eta} \left| z - \frac{x+y}{2} \right|^2 \\ &\quad + \tau u\left(\frac{x+y}{2}\right) - \frac{u(x) + u(y)}{2} \\ &\leq \sqrt{n}\tau \left| z - \frac{x+y}{2} \right| - \frac{1}{2\eta} \left| z - \frac{x+y}{2} \right|^2 \\ &\quad + \tau u\left(\frac{x+y}{2}\right) - \frac{u(x) + u(y)}{2}, \end{aligned}$$

From our arguments in part 1 above, it follows that

$$\lim_{|(x,y,z)| \rightarrow \infty} (w^\tau - \varphi_\eta)(x, y, z) = -\infty$$

and, in particular, that there is $(x_\eta, y_\eta, z_\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ maximizing $w^\tau - \varphi_\eta$. Now it is possible to argue analogously to how we did in Proposition 2.2 to conclude that $\mathcal{C}^\tau(x, y) \leq O(1 - \tau)$, as $\tau \rightarrow 1^-$. See also the proof of Lemma 3.7 in [18]. \square

Aleksandrov's Theorem (Theorem 1, page 242 [14]) now implies the following corollary.

Corollary 2.9. *u_δ is twice differentiable at (Lebesgue) almost every point in \mathbb{R}^n .*

Since u_δ is convex and $u_\delta \leq \bar{u}$ given by (2.9), we expect

$$\begin{aligned} \delta u_\delta - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u_\delta A + (x + ADu_\delta) \otimes (x + ADu_\delta)) &\leq \delta u_\delta - \frac{1}{2} \text{tr} \sigma \sigma^t A - \frac{1}{2} |\sigma^t (x + ADu_\delta)|^2 \\ &\leq K + \sum_{i=1}^n |x_i| - \frac{1}{2} \text{tr} \sigma \sigma^t A \\ &\quad - \frac{1}{2} (|\sigma^t x| - \sqrt{n}|A|)^2 \\ &< 0 \end{aligned}$$

for all x large enough and $\delta \in (0, 1)$, where K is the constant in (2.9). In other words, if $\max_{1 \leq i \leq n} |\partial_{x_i} u_\delta(x)| < 1$, then $|x| \leq C$ for some C independent of $\delta \in (0, 1)$. The appropriate statement in terms of jets is given below.

Corollary 2.10. *There is a constant $C = C(A) > 0$, independent of $0 < \delta < 1$, such that if $|x| \geq C$ and $p \in J^{1,-}u_\delta(x)$, then $\max_{1 \leq i \leq n} |p_i| \geq 1$.*

Proof. Choose $C = C(A)$ so large that

$$\delta u_\delta(z) < \frac{1}{2} \text{tr} \sigma \sigma^t A + \frac{1}{2} (|\sigma^t z| - \sqrt{n} |A|)^2, \quad |z| \geq C$$

for $0 < \delta < 1$. Recall that $J^{1,-}u_\delta(x) = \underline{\partial} u_\delta(x)$ by the convexity of u_δ (see Proposition 4.7 in [1]).⁵ Moreover, $(p, 0) \in J^{2,-}u_\delta(x)$, and so

$$\max_{1 \leq i \leq n} \left\{ \delta u_\delta(x) - \frac{1}{2} \text{tr} \sigma \sigma^t A - \frac{1}{2} |\sigma^t(x + Ap)|^2, |p_i| - 1 \right\} \geq 0.$$

As

$$\delta u_\delta(x) - \frac{1}{2} \text{tr} \sigma \sigma^t A - \frac{1}{2} |\sigma^t(x + Ap)|^2 \leq \delta u_\delta(x) - \frac{1}{2} \text{tr} \sigma \sigma^t A - \frac{1}{2} (|\sigma^t x| - \sqrt{n} |A|)^2 < 0,$$

$$\max_{1 \leq i \leq n} |p_i| \geq 1. \quad \square$$

Corollary 2.11. *There is a constant $C = C(A) > 0$, independent of $0 < \delta < 1$, such that*

$$u_\delta(x) = \min_{|y| \leq C} \left\{ u_\delta(y) + \sum_{i=1}^n |x_i - y_i| \right\}, \quad x \in \mathbb{R}^n. \quad (2.11)$$

Proof. Choose $C = C(A)$ such that

$$K + \sum_{i=1}^n |x_i| - \frac{1}{2} \text{tr} \sigma \sigma^t A - \frac{1}{2} (|\sigma^t x| - \sqrt{n} |A|)^2 \leq 0 \quad \text{for } |x| \geq C,$$

where K is the constant appearing in the definition of \bar{u} in equation (2.9). Also set v to be the right hand side of (2.11). As $\max_{1 \leq i \leq n} |\partial_{x_i} u_\delta| \leq 1$

$$u_\delta \leq v$$

and $v = u_\delta$ for $|x| \leq C$.

It is clear that $\max_{1 \leq i \leq n} |v_{x_i}| \leq 1$, and it is also straightforward to verify that as u_δ is convex, v is convex, as well. Now let $(p, X) \in J^{2,+}v(x_0)$. If $|x_0| < C$, the $v = u_\delta$ is a neighborhood of x_0 and so

$$\max_{1 \leq i \leq n} \left\{ \delta v(x_0) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)), |p_i| - 1 \right\} \leq 0.$$

⁵ $\underline{\partial} v(x) := \{p \in \mathbb{R}^n : v(y) \geq v(x) + p \cdot (y - x), \text{ for all } y \in \mathbb{R}^n\}$

If $|x_0| \geq C$, then by the convexity of v

$$\begin{aligned}
\delta v(x_0) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)) &\leq \delta v(x_0) - \frac{1}{2} \text{tr} \sigma \sigma^t A - |\sigma^t(x_0 + ADp)|^2 \\
&\leq \delta \left(u(0) + \sum_{i=1}^n |x_0 \cdot e_i| \right) - \frac{1}{2} \text{tr} \sigma \sigma^t A \\
&\quad - \frac{1}{2} (|\sigma^t x_0| - \sqrt{n}|A|)^2 \\
&\leq K + \sum_{i=1}^n |x_0 \cdot e_i| - \frac{1}{2} \text{tr} \sigma \sigma^t A \\
&\quad - \frac{1}{2} (|\sigma^t x_0| - \sqrt{n}|A|)^2 \\
&\leq 0,
\end{aligned}$$

while we always have $\max_{1 \leq i \leq n} |p_i| \leq 1$. Therefore, v is a subsolution of (2.7), and consequently

$$v \leq u_\delta.$$

□

Towards establishing an important lower bound on u_δ , we first observe that u_δ has its global minimum value at $x = 0$.

Proposition 2.12. $0 \in \partial u_\delta(0)$, and in particular u_δ achieves its minimum value at $x = 0$.

Proof. By Theorem 2.7,

$$u_\delta(x) = u_\delta(-x), \quad x \in \mathbb{R}^n$$

as $x \mapsto u_\delta(-x)$ satisfies (2.7) and (1.7). If u_δ is differentiable at $x = 0$, then $Du_\delta(0) = 0$. By convexity,

$$u_\delta(x) \geq u_\delta(0) + Du_\delta(0) \cdot x = u_\delta(0), \quad x \in \mathbb{R}^n.$$

In general (not assuming differentiability at $x = 0$), we write $u = u_\delta$ and set

$$u^\epsilon(x) = \eta^\epsilon * u(x) = \int_{\mathbb{R}^n} \eta^\epsilon(y) u(x - y) dy, \quad x \in \mathbb{R}^n \quad (2.12)$$

where $\eta^\epsilon \in C^\infty$ is the standard mollifier (see Appendix C of [8] for more on mollifiers). Clearly $u^\epsilon \in C^\infty(\mathbb{R}^n)$. Recall η^ϵ is radially symmetric, is supported in the ball B_ϵ , and satisfies $\int \eta^\epsilon = 1$ for all $\epsilon > 0$. As u is continuous, $u^\epsilon \rightarrow u$ locally uniformly as $\epsilon \rightarrow 0^+$. One checks that u^ϵ is convex and also that $u^\epsilon(x) = u^\epsilon(-x)$. From our remarks above, we conclude $u^\epsilon(x) \geq u^\epsilon(0)$ for $x \in \mathbb{R}^n$. Sending $\epsilon \rightarrow 0^+$, gives $u(x) \geq u(0)$ for all $x \in \mathbb{R}^n$. □

We conclude this subsection by establishing an crucial lower bound on u_δ ; this lower bound is key to establishing the existence of an eigenvalue.

Corollary 2.13. *There is a constant $C = C(A) > 0$, independent of $0 < \delta < 1$, such that*

$$u_\delta(x) \geq u_\delta(0) + \left(\sum_{i=1}^n |x_i| - C \right)^+, \quad x \in \mathbb{R}^n.$$

Proof. By above proposition $u_\delta(x) \geq u_\delta(0)$ for all $x \in \mathbb{R}^n$ and so the claim follows directly from Corollary 2.11. \square

2.3 Existence

We assume that A is a fixed symmetric, $n \times n$ matrix and will now establish the existence of a unique eigenvalue $\lambda(A)$. Proposition 2.2 asserts uniqueness, so all that is left to prove is the existence of an eigenvalue. To this end, we will use the estimates we have obtained on the sequence of solutions u_δ :

$$\begin{cases} (\sum_{i=1}^n |x_i| - K)^+ + \frac{\text{tr} \sigma \sigma^t A}{2\delta} \leq u_\delta(x) \leq \frac{K}{\delta} + \sum_{i=1}^n |x_i| \\ |u_\delta(x) - u_\delta(y)| \leq \sum_{i=1}^n |x_i - y_i| \\ u_\delta((x+y)/2) \leq (u_\delta(x) + u_\delta(y))/2, \end{cases}$$

for $x, y \in \mathbb{R}^n$ and $0 < \delta < 1$.

Define

$$\begin{cases} \lambda_\delta := \delta u_\delta(0) \\ v_\delta(x) := u_\delta(x) - u_\delta(0) \end{cases}.$$

Notice that

$$\frac{1}{2} \text{tr} \sigma \sigma^t A \leq \lambda_\delta \leq K$$

and v_δ satisfies

$$\begin{cases} |v_\delta(x)| \leq \sum_{i=1}^n |x_i| \\ |v_\delta(x) - v_\delta(y)| \leq \sum_{i=1}^n |x_i - y_i| \end{cases}$$

for $x \in \mathbb{R}^n$. We are now in position to prove the existence of an eigenvalue.

Lemma 2.14. *There is a sequence $\delta_k > 0$ tending to 0 as $k \rightarrow \infty$, $\lambda(A) \in \mathbb{R}$, and $u \in C(\mathbb{R}^n)$ with $|u(x) - u(y)| \leq \sum_{i=1}^n |x_i - y_i|$ such that*

$$\begin{cases} \lambda(A) = \lim_{k \rightarrow \infty} \lambda_{\delta_k} \\ v_{\delta_k} \rightarrow u \text{ in locally uniformly as } k \rightarrow \infty \end{cases}. \quad (2.13)$$

Moreover, u is a convex solution of (1.6) with eigenvalue $\lambda(A)$ that satisfies the growth condition (1.7).

Proof. It is immediate that $\lambda(A) = \lim_{k \rightarrow \infty} \lambda_{\delta_k}$ for some $\delta_k \rightarrow 0$, as λ_δ is bounded. The convergence assertion of a subsequence v_{δ_k} to some u , locally uniformly in \mathbb{R}^n , follows from the Arzelà-Ascoli theorem and a routine diagonalization argument; it is clear $|u(x) - u(y)| \leq \sum_{i=1}^n |x_i - y_i|$ and that u is convex. It also follows easily from the convergence assertion and the stability properties of viscosity solutions (Lemma 6.1 of [6]) that u satisfies the PDE

$$\max_{1 \leq i \leq n} \left\{ \lambda(A) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u A + (x + ADu) \otimes (x + ADu)), |u_{x_i}| - 1 \right\} = 0, \quad x \in \mathbb{R}^n$$

(in the sense of viscosity solutions). As $|u(x)| \leq \sum_{i=1}^n |x_i|$ for all $x \in \mathbb{R}^n$,

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} \leq 1.$$

By Corollary 2.13, for all $|x|$ sufficiently large

$$v_\delta(x) = u_\delta(x) - u_\delta(0) \geq \sum_{i=1}^n |x_i| - C,$$

for some C independent of $0 < \delta < 1$. Thus,

$$\liminf_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} \geq 1,$$

and so u satisfies (1.7). □

To complete the proof of Theorem 1.1, we establish the following regularity assertion.

Proposition 2.15. *Let u be as described in the statement of Lemma (2.14). If $\det A \neq 0$, then $u \in C_{loc}^{1,\alpha}(\mathbb{R}^n)$ for each $0 < \alpha < 1$.*

Proof. Note that u is a viscosity solution of the PDE

$$\max \{ Lu - b(x, Du), H(Du) \} = 0, \quad x \in \mathbb{R}^n, \tag{2.14}$$

where

$$\begin{cases} L\psi := -\frac{1}{2} \text{tr} \sigma \sigma^t AD^2 \psi A & \text{is uniformly elliptic, as } \det A \neq 0 \\ b(x, p) := -\lambda(A) + \frac{1}{2} \text{tr} \sigma \sigma^t A + \frac{1}{2} |\sigma^t(x + Ap)|^2 & \text{depends quadratically on } p \\ H(p) := \max_{1 \leq i \leq n} |p_i| - 1 & \text{is convex} \end{cases}.$$

Now let $O \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and consider the PDE

$$\begin{cases} Lu^\epsilon + \beta_\epsilon(H(Du^\epsilon)) = b(x, Du^\epsilon), & x \in O \\ u^\epsilon = u, & x \in \partial O \end{cases},$$

where $(\beta_\epsilon)_{\epsilon>0}$ is a family of functions $(\beta_\epsilon)_{\epsilon>0}$ satisfying

$$\begin{cases} \beta_\epsilon \in C^\infty(\mathbb{R}) \\ \beta_\epsilon = 0, \quad z \leq 0 \\ \beta_\epsilon > 0, \quad z > 0 \\ \beta'_\epsilon \geq 0 \\ \beta''_\epsilon \geq 0 \\ \beta_\epsilon(z) = \frac{z-\epsilon}{\epsilon}, \quad z \geq 2\epsilon \end{cases} . \quad (2.15)$$

For each $\epsilon > 0$, we think of β_ϵ as a type of smoothing of $z \mapsto (z/\epsilon)^+$; for small ϵ , we think of β_ϵ as a smooth approximation of the set valued mapping

$$\beta_0(t) = \begin{cases} \{0\}, & t < 0 \\ [0, \infty], & t = 0 \end{cases} .$$

Since the values of $\beta_\epsilon(H(Du^\epsilon))$ can be large when $H(Du^\epsilon) > 0$ and ϵ small, solutions will seek to satisfy $H(Du^\epsilon) \leq 0$ and, in this sense, become closer to satisfying equation (2.14). In this sense, solutions u^ϵ of (2.14) approximate u .

Theorem 15.10 in [16] implies that has a unique classical solution u^ϵ of equation (2.14); the crucial hypothesis here is that b grows at most quadratically in p . A minor modification of proof of Theorem 1.1 in [17] can be used to show that for each $O' \subset\subset O$ and $\alpha \in (0, 1)$ there is a constant $C = C(O', \alpha)$ such that

$$|u^\epsilon|_{C^{1,\alpha}(O')} \leq C,$$

and moreover that there is a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ tending to 0 such that $u^{\epsilon_k} \rightarrow u$ in $C^{1,\alpha}_{\text{loc}}(O)$, as $k \rightarrow \infty$. The important assumption here is that H is convex. From whence it follows that $u \in C^{1,\alpha}_{\text{loc}}(O)$ and since O was arbitrary, $u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$. \square

Corollary 2.16. *Let u be as described in the statement of Lemma (2.14). Then*

$$\Omega := \left\{ x \in \mathbb{R}^n : \max_{1 \leq i \leq n} |u_{x_i}(x)| < 1 \right\}$$

is open and bounded. Moreover, $u \in C^\infty(\Omega)$

Proof. The first assertion follows immediately from the previous proposition and Corollary 2.9, since $x \mapsto Du(x)$ is continuous mapping of \mathbb{R}^n into itself. The second assertion follows from standard elliptic regularity, as u satisfies a semilinear elliptic PDE on Ω (see Theorem 6.17 [16]). \square

3 Properties of the eigenvalue function

In view of Theorem 1.1, the solution of the eigenvalue problem defines a function that we shall denote $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$. In this section, we prove Theorem 1.2, which details some important properties of this function. Our basic tool will be the comparison principle described in Proposition 2.2. We use this property to show that λ is a monotone, convex function. Moreover, the regularity result of the previous section will be used to establish minmax formulae for λ . In order to establish these properties, we will make use of the following characterizations of λ , which follow immediately from the existence and uniqueness of the eigenvalue function.

Proposition 3.1. *Let $A \in \mathcal{S}(n)$ and assume that $\lambda(A)$ is the solution of the eigenvalue problem associated with equation (1.6). Then*

$$\lambda(A) = \sup \left\{ \lambda \in \mathbb{R} : \text{there exists a subsolution } u \text{ of (1.6) with eigenvalue } \lambda, \right. \\ \left. \text{satisfying } \limsup_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} \leq 1. \right\} \quad (3.1)$$

and

$$\lambda(A) = \inf \left\{ \mu \in \mathbb{R} : \text{there exists a supersolution } v \text{ of (1.6) with eigenvalue } \mu, \right. \\ \left. \text{satisfying } \liminf_{|x| \rightarrow \infty} \frac{v(x)}{\sum_{i=1}^n |x_i|} \geq 1. \right\}. \quad (3.2)$$

The above formulae, manifestations of the comparison principle, will be used below to establish monotone upper and lower bounds on the eigenvalue that will be crucial to deduce other properties. Before pursuing these bounds, we note a basic *symmetry* of λ . The following proposition states that $\lambda(A)$ is invariant under permutations of A .

Proposition 3.2. *For any $A \in \mathcal{S}(n)$ and permutation matrix U , we have*

$$\lambda(UAU^t) = \lambda(A).$$

Proof. Let $A \in \mathcal{S}(n)$ and $u(\cdot; A)$ be a solution of (1.6) with eigenvalue $\lambda(A)$. Direct computation has that

$$v(x; UAU^t) := u(U^t x; A)$$

is also a solution of (1.6) that satisfies (1.7) for any permutation matrix U ; the key observation here is that $v_{x_i} = Du(U^t x) \cdot U^t e_i$ and U^t permutes the standard basis vectors. By Proposition 2.2, we have $\lambda(A) = \lambda(UAU^t)$. \square

3.1 Monotone upper and lower bounds

In this subsection, we prove that λ is a locally bounded, nondecreasing, convex function and therefore it is necessarily continuous. We first show that the function λ is bounded above and below by monotone functions that are constructed from $\lambda_1 : \mathbb{R} \rightarrow \mathbb{R}$, the solution of the eigenvalue problem found by G. Barles and H. Soner [2]. Then we show λ is convex by an elementary argument. It turns out that any convex function that is bounded above by a nondecreasing function is necessarily nondecreasing itself, and therefore we will be able to conclude that λ is monotone. This implies, in particular, that the PDE $\psi_t + \lambda(d(p)D^2\psi d(p)) = 0$ is backwards parabolic which will be useful to us in the following section.

Proposition 3.3. *There are monotone non-decreasing functions $\underline{\lambda}, \bar{\lambda} : \mathcal{S}(n) \rightarrow \mathbb{R}$ such that*

$$\underline{\lambda}(A) \leq \lambda(A) \leq \bar{\lambda}(A),$$

for all $A \in \mathcal{S}(n)$.

In dimension $n = 1$, G. Barles and H. Soner [2] showed that the eigenvalue problem associated to the ODE

$$\max \left\{ \lambda - \frac{\sigma^2}{2}(A + A^2 u'' + (x + Au')^2), |u'| - 1 \right\} = 0, \quad x \in \mathbb{R} \quad (3.3)$$

has a unique solution $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ that is monotone increasing and continuous. Moreover, associated to $\lambda(A)$ is a solution $u = u(\cdot ; A) \in C(\mathbb{R})$ that satisfies

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1.$$

Furthermore, when $A \neq 0$, $u(\cdot ; A) \in C^2(\mathbb{R})$. We will need the following variant of this result for our purposes.

Lemma 3.4. *(solution of the 1D eigenvalue problem)*

(i) *For each $A \in \mathbb{R}$ and $\alpha > 0$, there is a unique $\lambda = \lambda_1(A, \alpha) \in \mathbb{R}$ such that the ODE*

$$\max \left\{ \lambda - \frac{\sigma^2}{2}(A + A^2 u'' + (x + Au')^2), |u'| - \alpha \right\} = 0, \quad x \in \mathbb{R} \quad (3.4)$$

has a solution $u = u_1(\cdot ; A, \alpha) \in C(\mathbb{R})$ satisfying

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = \alpha. \quad (3.5)$$

When $A \neq 0$, $u_1(\cdot ; A, \alpha) \in C^2(\mathbb{R})$.

(ii) *The function $A \mapsto \lambda_1(A, \alpha)$ is continuous and monotone non-decreasing for each $\alpha > 0$.*

Proof. Let $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ be the solution of the eigenvalue problem associated to the ODE (3.3), with solution $u = u(\cdot; A)$ for each $A \in \mathbb{R}$ as described above. It is easy to check that

$$u_1(x; A, \alpha) := u(\alpha x; \alpha^2 A), \quad x \in \mathbb{R}$$

is a solution of (3.4) with eigenvalue

$$\lambda_1(A, \alpha) := \frac{\lambda(\alpha^2 A)}{\alpha^2}$$

that satisfies (3.5). The uniqueness of λ_1 follows from the same ideas used to prove Proposition 2.2. \square

We shall use λ_1 to design $\underline{\lambda}$ and $\bar{\lambda}$ in Proposition 3.3. Our main tool will be formulae (3.1) and (3.2). First, however, we will perform a change of variables and rewrite (1.6). For a given $A \in \mathcal{S}(n)$, we may write

$$\sigma^t A \sigma = P \Lambda P^t$$

where $P^t P = I_n$ and

$$\Lambda = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}. \quad (3.6)$$

Note that

$$\text{tr} \sigma^t A \sigma = \sum_{i=1}^n a_i,$$

and for

$$v(z) = u(x) \quad \text{with} \quad z = \sigma^t x,$$

$$\begin{aligned} \text{tr} \sigma^t \sigma A D^2 u A &= \text{tr}(\sigma^t A \sigma) D^2 v(\sigma^t A \sigma) \\ &= \text{tr}[P \Lambda P^t D^2 v P \Lambda P^t] \\ &= \text{tr}[\Lambda P^t D^2 v P \Lambda] \\ &= \text{tr}[\Lambda^2 P^t D^2 v P] \\ &= \sum_{i=1}^n a_i^2 (P^t D^2 v P) e_i \cdot e_i \\ &= \sum_{i=1}^n a_i^2 D^2 v P e_i \cdot P e_i \\ &= \sum_{i=1}^n a_i^2 v_{P e_i, P e_i}, \end{aligned}$$

and

$$\begin{aligned}
\text{tr} \sigma \sigma^t (x + ADu) \otimes (x + ADu) &= |\sigma^t (x + ADu)|^2 \\
&= |\sigma^t x + \sigma^t ADu|^2 \\
&= |z + (\sigma^t A \sigma) Dv|^2 \\
&= |PP^t z + P \Lambda P^t Dv|^2 \\
&= |P^t z + \Lambda P^t Dv|^2 \\
&= \sum_{i=1}^n (z \cdot P e_i + a_i v_{P e_i})^2.
\end{aligned}$$

Making a further change of variables

$$w(y) = v(z) \quad \text{with} \quad y = P^t z,$$

we have that if

$$\max_{1 \leq i \leq n} \left\{ \lambda - \frac{1}{2} \text{tr} \sigma \sigma^t [A + AD^2 u A + (x + ADu) \otimes (x + ADu)], |u_{x_i}| - 1 \right\} = 0, \quad x \in \mathbb{R}^n$$

then

$$\max_{1 \leq i \leq n} \left\{ \lambda - \frac{1}{2} \sum_{i=1}^n (a_i + a_i^2 w_{y_i y_i} + (y_i + a_i w_{y_i})^2), |\sigma P D w \cdot e_i| - 1 \right\} = 0, \quad y \in \mathbb{R}^n.$$

The above PDE is closely related to the equation

$$\max_{1 \leq i \leq n} \left\{ \lambda - \frac{1}{2} \sum_{i=1}^n (a_i + a_i^2 w_{y_i y_i} + (y_i + a_i w_{y_i})^2), |w_{y_i}| - 1 \right\} = 0, \quad y \in \mathbb{R}^n,$$

which has the *separation of variables* solution

$$\lambda = \sum_{i=1}^n \lambda_1(a_i, 1) \quad \text{and} \quad w(y) = \sum_{i=1}^n u_1(y_i; a_i, 1), \quad y \in \mathbb{R}^n$$

(here and below, λ_1 and u_1 are a solution pair of (3.3) with $\sigma^2 = 1$). These computations motivate the following lemma.

Lemma 3.5. *Let $A \in \mathcal{S}(n)$ and assume that $\sigma^t A \sigma = P \Lambda P^t$ where $P^t P = I_n$ and Λ is given by (3.6).*

(i) Set

$$\underline{\lambda} := \sum_{i=1}^n \lambda_1(a_i, 1/|\sigma|\sqrt{n}) \quad \text{and} \quad \underline{w}(y) := \sum_{i=1}^n u_1(y_i; a_i, 1/|\sigma|\sqrt{n})$$

where λ_1 and u_1 are the solutions of (3.4) as described in Lemma 3.4. Then $\underline{u} : x \mapsto \underline{w}((\sigma P)^t x)$ is a subsolution of (1.6) with eigenvalue $\underline{\lambda}$.

(ii) Set

$$\bar{\lambda} := \sum_{i=1}^n \lambda_1(a_i, |\sigma^{-1}| \sqrt{n}) \quad \text{and} \quad \bar{w}(y) := \sum_{i=1}^n u_1(y_i; a_i, |\sigma^{-1}| \sqrt{n})$$

where λ_1 and u_1 are the solutions of (3.4) as described in Lemma 3.4. Then $\bar{u} : x \mapsto \bar{w}((\sigma P)^t x)$ is a supersolution of (1.6) with eigenvalue $\bar{\lambda}$ satisfying

$$\liminf_{|x| \rightarrow \infty} \frac{\bar{u}(x)}{\sum_{i=1}^n |x_i|} \geq 1. \quad (3.7)$$

Proof. We prove the case where $\det A \neq 0$, so that $u_1(\cdot; a_i, \cdot) \in C^2(\mathbb{R})$ for $i = 1, \dots, n$. The general case then follows by straightforward limiting arguments and the stability of viscosity solutions under local uniform convergence.

(i) By assumption, \underline{u} is a solution of the equation

$$\max_{1 \leq i \leq n} \left\{ \underline{\lambda} - \frac{1}{2} \sum_{i=1}^n (a_i + a_i^2 w_{y_i y_i} + (y_i + a_i w_{y_i})^2), |w_{y_i}| - \frac{1}{|\sigma| \sqrt{n}} \right\} = 0, \quad y \in \mathbb{R}^n.$$

As

$$\text{tr} \sigma \sigma^t [A + AD^2 \underline{u} A + (x + AD \underline{u}) \otimes (x + AD \underline{u})] = \sum_{i=1}^n (a_i + a_i^2 \underline{w}_{y_i y_i} + (y_i + a_i \underline{w}_{y_i})^2),$$

and

$$\begin{aligned} \max_{1 \leq i \leq n} |\underline{u}_{x_i}| &= \max_{1 \leq i \leq n} |\sigma P D \underline{w} \cdot e_i| \\ &\leq |\sigma P| |D \underline{w}| \\ &\leq |\sigma| |D \underline{w}| \\ &\leq |\sigma| \sqrt{n} \max_{1 \leq i \leq n} |\underline{w}_{y_i}| \\ &\leq 1, \end{aligned}$$

we have that

$$\max_{1 \leq i \leq n} \left\{ \underline{\lambda} - \frac{1}{2} \text{tr} \sigma \sigma^t [A + AD^2 \underline{u} A + (x + AD \underline{u}) \otimes (x + AD \underline{u})], |\underline{u}_{x_i}| - 1 \right\} \leq 0, \quad x \in \mathbb{R}^n.$$

Thus \underline{u} is a subsolution of (1.6) with eigenvalue $\underline{\lambda}$.

(ii) By assumption, \bar{w} is a solution of the equation

$$\max_{1 \leq i \leq n} \left\{ \bar{\lambda} - \frac{1}{2} \sum_{i=1}^n (a_i + a_i^2 w_{y_i y_i} + (y_i + a_i w_{y_i})^2), |w_{y_i}| - |\sigma^{-1}| \sqrt{n} \right\} = 0, \quad y \in \mathbb{R}^n.$$

Notice that

$$\text{tr} \sigma \sigma^t [A + AD^2 \bar{u} A + (x + AD \bar{u}) \otimes (x + AD \bar{u})] = \sum_{i=1}^n (a_i + a_i^2 \bar{w}_{y_i y_i} + (y_i + a_i \bar{w}_{y_i})^2)$$

and

$$\begin{aligned} \max_{1 \leq i \leq n} |\bar{u}_{x_i}| &= \max_{1 \leq i \leq n} |\sigma P D \bar{w} \cdot e_i| \\ &\geq \frac{1}{\sqrt{n}} |\sigma P D \bar{w}| \\ &\geq \frac{1}{\sqrt{n}} \frac{|D \bar{w}|}{|(\sigma P)^{-1}|} \\ &= \frac{1}{\sqrt{n}} \frac{|D \bar{w}|}{|P^t \sigma^{-1}|} \\ &\geq \frac{1}{\sqrt{n}} \frac{|D \bar{w}|}{|\sigma^{-1}|} \\ &\geq \frac{1}{\sqrt{n}} \frac{\max_{1 \leq i \leq n} |\bar{w}_{y_i}|}{|\sigma^{-1}|}, \end{aligned}$$

which of course implies

$$|\sigma^{-1}| \sqrt{n} \left(\max_{1 \leq i \leq n} |\bar{u}_{x_i}| - 1 \right) \geq \max_{1 \leq i \leq n} |\bar{w}_{y_i}| - |\sigma^{-1}| \sqrt{n}.$$

It follows that

$$\max_{1 \leq i \leq n} \left\{ \bar{\lambda} - \frac{1}{2} \text{tr} \sigma \sigma^t [A + AD^2 \bar{u} A + (x + AD \bar{u}) \otimes (x + AD \bar{u})], |\bar{u}_{x_i}| - 1 \right\} \geq 0, \quad x \in \mathbb{R}^n.$$

Since,

$$\frac{\bar{u}(x)}{\sum_{i=1}^n |x_i|} \geq \frac{\sum_{i=1}^n u_1(y_i; a_i, |\sigma^{-1}| \sqrt{n})}{|\sigma^{-1}| \sqrt{n} \max_{1 \leq i \leq n} |y_i|},$$

where $y = (\sigma P)^t x$, and

$$\lim_{|t| \rightarrow \infty} \frac{u_1(t; a_i, |\sigma^{-1}| \sqrt{n})}{|t|} = |\sigma^{-1}| \sqrt{n},$$

we have

$$\liminf_{|x| \rightarrow \infty} \frac{\bar{u}(x)}{\sum_{i=1}^n |x_i|} \geq 1.$$

Hence, \bar{u} is a supersolution of (1.6) with eigenvalue $\bar{\lambda}$ that satisfies (3.7). \square

Corollary 3.6. *Define*

$$\underline{\lambda}(A) := \text{tr} \lambda_1(\sigma^t A \sigma, 1/|\sigma| \sqrt{n})$$

and

$$\bar{\lambda}(A) := \text{tr} \lambda_1(\sigma^t A \sigma, |\sigma^{-1}| \sqrt{n}).$$

for $A \in \mathcal{S}(n)$. Then

$$\underline{\lambda} \leq \lambda \leq \bar{\lambda}.$$

Proof. For $\sigma^t A \sigma = P \Lambda P^t$ where $P^t P = I_n$ and Λ is given by (3.6), we have

$$\text{tr} \lambda_1(\sigma^t A \sigma, 1/|\sigma| \sqrt{n}) = \sum_{i=1}^n \lambda_1(a_i, 1/|\sigma| \sqrt{n})$$

and

$$\text{tr} \lambda_1(\sigma^t A \sigma, |\sigma^{-1}| \sqrt{n}) = \sum_{i=1}^n \lambda_1(a_i, |\sigma^{-1}| \sqrt{n}).$$

Therefore, this corollary follows directly from the above lemma and formulae (3.1) and (3.2). \square

Proposition 3.3 is now established by noting that

$$A \mapsto \text{tr} \lambda_1(\sigma^t A \sigma, 1/|\sigma| \sqrt{n}) \quad \text{and} \quad A \mapsto \text{tr} \lambda_1(\sigma^t A \sigma, |\sigma| \sqrt{n})$$

are nondecreasing. This follows from the proposition below, which is proved in Appendix A.

Proposition 3.7. *Let $f \in C(\mathbb{R})$ be monotone non-decreasing. Then $\mathcal{S}(n) \ni A \mapsto \text{tr} f(A)$ is monotone non-decreasing with respect to the partial ordering on $\mathcal{S}(n)$.*

Now we turn to the regularity properties of λ and show λ is convex and necessarily continuous. As we mentioned above, this fact will be used to show that λ is monotone nondecreasing.

Proposition 3.8. $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$ is convex.

Proof. 1. Let $A_1, A_2 \in \mathcal{S}(n)$ and set $A_3 := (A_1 + A_2)/2$. We will first show

$$\lambda_3 \leq \frac{\lambda_1 + \lambda_2}{2}.$$

where $\lambda_i := \lambda(A_i)$, $i = 1, 2, 3$. Let $u_i = u(\cdot; A_i)$ and assume $u_i \in C^2(\mathbb{R}^n)$. We shall only give a formal proof, as we now have sufficient experience making the type of argument given below rigorous with standard viscosity solutions methods. Finally, we also assume $\frac{1}{2} \sigma \sigma^t = I_n$. A simple inspection of the reasoning below will convince the reader that this can be done without any loss of generality.

Fix $\tau > 0$ and note that the function

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x_1, x_2) \mapsto \tau u_3(x_3) - \frac{u_1(x_1) + u_2(x_2)}{2}, \quad x_3 := (x_1 + x_2)/2$$

has a maximum on $\mathbb{R}^n \times \mathbb{R}^n$. For simplicity, we denote this point by (x_1, x_2) and suppress the τ dependence. As (x_1, x_2) is a maximizer

$$\tau Du_3(x_3) = Du_1(x_1) = Du_2(x_2) \quad (3.8)$$

and

$$\begin{bmatrix} \frac{\tau}{4}D^2u_3(x_3) - \frac{1}{2}D^2u_1(x_1) & \frac{\tau}{4}D^2u_3(x_3) \\ \frac{\tau}{4}D^2u_3(x_3) & \frac{\tau}{4}D^2u_3(x_3) - \frac{1}{2}D^2u_2(x_2) \end{bmatrix} \leq 0.$$

The above matrix inequality implies that

$$\frac{\tau}{4}D^2u_3(x_3)(\xi_1 + \xi_2) \cdot (\xi_1 + \xi_2) \leq \frac{1}{2}D^2u_1(x_1)\xi_1 \cdot \xi_1 + \frac{1}{2}D^2u_2(x_2)\xi_2 \cdot \xi_2.$$

for each $\xi_1, \xi_2 \in \mathbb{R}^n$. Therefore, Lemma B.1 (proved in Appendix B) implies the inequality

$$\text{tr} [A_3 \tau D^2u_3(x_3) A_3] \leq \frac{1}{2} \text{tr} [A_1 D^2u_1(x_1) A_1] + \frac{1}{2} \text{tr} [A_2 D^2u_2(x_2) A_2]. \quad (3.9)$$

2. As

$$|\partial_{x_i} u_1(x_1)| = |\partial_{x_i} u_2(x_2)| = \tau |\partial_{x_i} u_3(x_3)| \leq \tau < 1, \quad i = 1, \dots, n$$

we have

$$\lambda_i - \text{tr} [A_i + A_i D^2u_i(x_i) A_i + (x_i + A_i Du_i(x_i)) \otimes (x_i + A_i Du_i(x_i))] = 0, \quad i = 1, 2.$$

Therefore, using the inequality (3.9) gives

$$\begin{aligned} \tau \lambda_3 - \frac{\lambda_1 + \lambda_2}{2} &\leq (\tau - 1) \text{tr} A_3 + \tau |x_3 + A_3 Du_3(x_3)|^2 \\ &\quad - \frac{1}{2} |x_1 + A_1 Du_1(x_1)|^2 - \frac{1}{2} |x_2 + A_2 Du_2(x_2)|^2 \\ &= (\tau - 1) \text{tr} A_3 + \tau |x_3|^2 - \frac{1}{2} |x_1|^2 - \frac{1}{2} |x_2|^2 + \tau |A_3 Du_3(x_3)|^2 \\ &\quad - \frac{1}{2} |A_1 Du_1(x_1)|^2 - \frac{1}{2} |A_2 Du_2(x_2)|^2 + 2\tau x_3 \cdot A_3 Du_3(x_3) \\ &\quad - x_1 \cdot A_1 Du_1(x_1) - x_2 \cdot A_2 Du_2(x_2). \end{aligned}$$

Basic manipulations are

$$\tau |x_3|^2 - \frac{1}{2} |x_1|^2 - \frac{1}{2} |x_2|^2 = (\tau - 1) |x_3|^2 - \left| \frac{x_1 - x_2}{2} \right|^2,$$

and using the first order conditions (3.8) gives

$$\begin{aligned} \tau |A_3 Du_3(x_3)|^2 - \frac{1}{2} |A_1 Du_1(x_1)|^2 - \frac{1}{2} |A_2 Du_2(x_2)|^2 &= \tau(1 - \tau) \frac{|A_1 Du_3(x_3)|^2 + |A_2 Du_3(x_3)|^2}{2} \\ &\quad - \left| \tau \left(\frac{A_1 - A_2}{2} \right) Du_3(x_3) \right|^2 \end{aligned}$$

and

$$\begin{aligned} 2\tau x_3 \cdot A_3 Du_3(x_3) - x_1 \cdot A_1 Du_1(x_1) - x_2 \cdot A_2 Du_2(x_2) &= \left(\frac{A_1 - A_2}{2} \right) \left(\frac{x_2 - x_1}{2} \right) \cdot \tau Du_3(x_3) \\ &\leq \left| \tau \left(\frac{A_1 - A_2}{2} \right) Du_3(x_3) \right|^2 + \left| \frac{x_1 - x_2}{2} \right|^2. \end{aligned}$$

Combining the previous four inequalities lead us to

$$\tau \lambda_3 - \frac{\lambda_1 + \lambda_2}{2} \leq (\tau - 1) \text{tr} A_3 + (\tau - 1) |x_3|^2 + \tau(1 - \tau) \frac{|A_1 Du_3(x_3)|^2 + |A_2 Du_3(x_3)|^2}{2} \leq C(1 - \tau)$$

for some universal constant C , as $\max_i |\partial_{x_i} u_3(x_3)| \leq 1$. We conclude by letting $\tau \rightarrow 1^-$.

3. Finally, we remark that virtually the same steps can be used to show

$$\lambda(sA + (1 - s)B) \leq s\lambda(A) + (1 - s)\lambda(B),$$

for $A, B \in \mathcal{S}(n)$ and $0 \leq s \leq 1$. Therefore, the argument above which shows that λ is midpoint convex also shows λ is convex. \square

Corollary 3.9. $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$ is continuous.

Corollary 3.10. $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$ is monotone nondecreasing.

Proof. It suffices to verify the general assertion that if $f, g : \mathcal{S}(n) \rightarrow \mathbb{R}$ with

$$\begin{cases} g \text{ nondecreasing} \\ f \text{ convex, and} \\ f \leq g, \end{cases}$$

then f is nondecreasing (In our case, $f = \lambda$ and $g = \bar{\lambda}$).

Suppose that $Q \in \partial f(A_0) \neq \emptyset$; that is,

$$Q \cdot (A - A_0) + f(A_0) \leq f(A), \quad A \in \mathcal{S}(n). \quad (3.10)$$

We claim $Q \geq 0$. To see this, let $\xi \in \mathbb{R}^n$ and set

$$A := A_0 - t\xi \otimes \xi$$

for $t > 0$. As g is nondecreasing, substituting this A in (3.10) gives

$$-tQ\xi \cdot \xi + f(A_0) \leq f(A_0 - t\xi \otimes \xi) \leq g(A_0 - t\xi \otimes \xi) \leq g(A_0).$$

Clearly this inequality holds for all $t > 0$ if and only if $Q\xi \cdot \xi \geq 0$ i.e. $Q \geq 0$ and thus f is nondecreasing. \square

3.2 Min-max formulae

In this subsection, we prove two minmax formulae for λ . To this end, we make use of formulae (3.1) and (3.2) and the regularity of solutions of equation (1.6). We remark that, while these alternative characterizations of λ are interesting, we do not use them in the proof of Theorem 1.3. Therefore, this subsection can be omitted without any loss of continuity.

Proposition 3.11. *For $A \in \mathcal{S}(n)$, set*

$$\lambda_-(A) = \sup \left\{ \inf_{x \in \mathbb{R}^n} \frac{1}{2} \text{tr} \sigma \sigma^t \left(A + AD^2 \phi(x) A + (x + AD \phi(x)) \otimes (x + AD \phi(x)) \right) : \right. \\ \left. \phi \in C^2(\mathbb{R}^n), \max_{1 \leq i \leq n} |\phi_{x_i}| \leq 1 \right\}$$

and

$$\lambda_+(A) = \inf \left\{ \sup_{\max_i |\psi_{x_i}(x)| < 1} \frac{1}{2} \text{tr} \sigma \sigma^t \left(A + AD^2 \psi(x) A + (x + AD \psi(x)) \otimes (x + AD \psi(x)) \right) : \right. \\ \left. \psi \in C^2(\mathbb{R}^n), \liminf_{|x| \rightarrow \infty} \frac{\psi(x)}{\sum_{i=1}^n |x_i|} \geq 1 \right\}.$$

Then

$$\lambda_-(A) \leq \lambda(A) \leq \lambda_+(A).$$

Moreover, equality holds in both inequalities when $\det A \neq 0$.

Proof. ($\lambda_- \leq \lambda$) Fix $A \in \mathcal{S}(n)$, let $\phi \in C^2$ and suppose that $\max_i |\phi_{x_i}| \leq 1$. Now set

$$\mu^\phi(A) := \inf_{x \in \mathbb{R}^n} \frac{1}{2} \text{tr} \sigma \sigma^t \left(A + AD^2 \phi(x) A + (x + AD \phi(x)) \otimes (x + AD \phi(x)) \right).$$

If $\mu^\phi(A) = -\infty$, then $\mu^\phi(A) \leq \lambda(A)$; if $\mu^\phi(A) > -\infty$, by the assumptions on ϕ and the definition of $\mu^\phi(A)$

$$\max_{1 \leq i \leq n} \left\{ \mu^\phi(A) - \frac{1}{2} \text{tr} \sigma \sigma^t \left(A + AD^2 \phi A + (x + AD \phi) \otimes (x + AD \phi) \right), |\phi_{x_i}| - 1 \right\} \leq 0.$$

By (3.1), we still have $\mu^\phi(A) \leq \lambda(A)$. Thus,

$$\lambda_-(A) = \sup_{\phi} \mu^\phi(A) \leq \lambda(A).$$

($\lambda_+ \geq \lambda$) Again fix $A \in \mathcal{S}(n)$. Now let $\psi \in C^2$ satisfy $\liminf_{|x| \rightarrow \infty} \psi(x) / \sum_{i=1}^n |x_i| \geq 1$ and set

$$\tau^\psi(A) := \sup_{\max_i |\psi_{x_i}(x)| < 1} \frac{1}{2} \text{tr} \sigma \sigma^t \left(A + AD^2 \psi(x) A + (x + AD \psi(x)) \otimes (x + AD \psi(x)) \right).$$

If $\tau^\psi(A) = +\infty$, then $\tau^\psi(A) \geq \lambda(A)$; if $\tau^\psi(A) < +\infty$, by the assumptions on ψ and the definition of $\tau^\psi(A)$

$$\max \left\{ \tau^\psi(A) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 \psi A + (x + AD\psi) \otimes (x + AD\psi)), |\psi_{x_i}| - 1 \right\} \geq 0.$$

By (3.2), we still have $\tau^\psi(A) \geq \lambda(A)$. Hence,

$$\lambda_+(A) = \inf_{\psi} \tau^\psi(A) \geq \lambda(A).$$

(Equality) Suppose that $\det A \neq 0$ and let $u = u(\cdot, A)$ be a convex solution of (1.6) associated to $\lambda(A)$ that satisfies $u \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$ for each $\alpha \in (0, 1)$. We first claim that $\lambda(A) \leq \lambda_-(A)$. To see this we mollify u

$$u^\epsilon := \eta^\epsilon * u$$

(see (2.12) for more on mollification). It is a basic fact that if $f \in C_{\text{loc}}^\alpha(\mathbb{R}^n)$ then

$$|f^\epsilon(x) - f(x)| \leq |f|_{C^\alpha(O)} \epsilon^\alpha, \quad x \in O_\epsilon := \{x \in O : \text{dist}(x, \partial O) \geq \epsilon\}$$

for each bounded domain $O \subset \mathbb{R}^n$.

Using this simple observation of mollifiers of Hölder continuous functions on \mathbb{R}^n , the fact that $Du \in C_{\text{loc}}^\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and some elementary computations, we find

$$\eta^\epsilon * (x \mapsto |\sigma^t(x + ADu(x))|^2) = |\sigma^t(x + ADu^\epsilon)|^2 + O(\epsilon^\alpha)$$

as $\epsilon \rightarrow 0$, for x belonging to bounded subdomains of \mathbb{R}^n . Therefore, as u solves the PDE (1.6) almost everywhere on \mathbb{R}^n

$$\lambda(A) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u^\epsilon A + (x + ADu^\epsilon) \otimes (x + ADu^\epsilon)) \leq O(\epsilon^\alpha)$$

for x belonging to bounded subdomains of \mathbb{R}^n .

Also notice that as u^ϵ is convex and Du^ϵ is uniformly bounded, the function

$$\mathbb{R}^n \ni x \mapsto \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u^\epsilon A + (x + ADu^\epsilon) \otimes (x + ADu^\epsilon))$$

has a minimum in a ball $B_R(0)$ for some R that is *independent* of $\epsilon > 0$. Consequently,

$$\begin{aligned} \lambda_-(A) &\geq \inf_{\mathbb{R}^n} \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u^\epsilon A + (x + ADu^\epsilon) \otimes (x + ADu^\epsilon)) \\ &= \inf_{B_R} \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u^\epsilon A + (x + ADu^\epsilon) \otimes (x + ADu^\epsilon)) \\ &\geq \lambda(A) + O(\epsilon^\alpha). \end{aligned}$$

Next, we claim that $\lambda(A) \geq \lambda_+(A)$. An important observation for us will be that $\max_{1 \leq i \leq n} |u_{x_i}|$ is uniformly continuous on \mathbb{R}^n . This is due to

$$\lim_{|x| \rightarrow \infty} \max_{1 \leq i \leq n} |u_{x_i}(x)| = 1,$$

which in turn follows from the limit (1.7) and the fact that u is convex. An immediate corollary of this uniform convergence is that Du^ϵ converges to Du *uniformly* on \mathbb{R}^n , where $u^\epsilon = \eta^\epsilon * u$.

Set

$$u^{\epsilon, \delta} := (1 + \delta)u^\epsilon,$$

where $\delta > 0$ is fixed, and notice that

$$\max_{1 \leq i \leq n} |u_{x_i}^{\epsilon, \delta}(x)| < 1 \Leftrightarrow \max_{1 \leq i \leq n} |u_{x_i}^\epsilon(x)| < \frac{1}{1 + \delta}.$$

As $1/(1 + \delta) < 1$, there is $\rho = \rho(\delta) > 0$ so small such that

$$\gamma := \frac{1}{1 + \delta} + \rho < 1.$$

Also, for $\epsilon_0 = \epsilon_0(\delta) > 0$ small enough

$$\max_{1 \leq i \leq n} |u_{x_i}(x)| \leq \max_{1 \leq i \leq n} |u_{x_i}^\epsilon(x)| + \rho < \frac{1}{1 + \delta} + \rho = \gamma,$$

provided $0 < \epsilon < \epsilon_0$ and $\max_{1 \leq i \leq n} |u_{x_i}^{\epsilon, \delta}(x)| < 1$. Moreover, there is $\epsilon_1 = \epsilon_1(\delta)$ such that

$$\{x \in \mathbb{R}^n : \max_{1 \leq i \leq n} |u_{x_i}(x)| < \gamma\} \subset \Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$$

for $0 < \epsilon < \epsilon_1$. This inclusion follows as the set $\{x \in \mathbb{R}^n : \max_{1 \leq i \leq n} |u_{x_i}(x)| < \gamma\}$ is an open subset of Ω .

Hence for $0 < \epsilon < \min\{\epsilon_0, \epsilon_1\}$, we have

$$\max_{1 \leq i \leq n} |u_{x_i}^{\epsilon, \delta}(x)| < 1 \Rightarrow x \in \Omega_\epsilon.$$

In particular, if $\max_{1 \leq i \leq n} |u_{x_i}^{\epsilon, \delta}(x)| < 1$, then

$$\lambda(A) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u^\epsilon A + (x + ADu^\epsilon) \otimes (x + ADu^\epsilon)) = O(\epsilon^\alpha),$$

as $\lambda(A) - \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u A + (x + ADu) \otimes (x + ADu)) = 0$, a.e. on Ω .

With the above computations and the fact that $u \in C^\infty(\Omega)$ we have

$$\begin{aligned} \lambda_+(A) &\leq \sup_{\max_i |u_{x_i}^{\epsilon, \delta}(x)| < 1} \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u^{\epsilon, \delta} A + (x + ADu^{\epsilon, \delta}) \otimes (x + ADu^{\epsilon, \delta})) \\ &= \sup_{\max_i |u_{x_i}^{\epsilon, \delta}(x)| < 1} \frac{1}{2} \text{tr} \sigma \sigma^t (A + AD^2 u^\epsilon A + (x + ADu^\epsilon) \otimes (x + ADu^\epsilon)) + O(\delta) \\ &= \lambda(A) + O(\epsilon^\alpha) + O(\delta). \end{aligned}$$

We conclude by first sending $\epsilon \rightarrow 0^+$ and then $\delta \rightarrow 0^+$. □

Conjecture 3.12. $\lambda_-(A) = \lambda(A) = \lambda_+(A)$ for all $A \in \mathcal{S}(n)$.

Remark 3.13. The conjecture would follow from additional properties of λ_\pm such as monotonicity, convexity, or continuity.

4 Convergence

In this section, we verify Theorem 1.3 which characterizes $\lim_{\epsilon \rightarrow 0^+} z^\epsilon$ as a solution of the nonlinear diffusion equation

$$\psi_t + \lambda(d(p)D^2\psi d(p)) = 0, \quad (t, p) \in (0, T) \times (0, \infty)^n. \quad (4.1)$$

Here, $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$ is of course the solution of the eigenvalue problem discussed in previous sections. The method of proof is relatively standard in the theory of viscosity solutions and goes as follows. We show the upper limit

$$\bar{z}(t, p, y) := \limsup_{\substack{\epsilon \rightarrow 0^+ \\ (t', p', y') \rightarrow (t, p, y)}} z^\epsilon(t', p', y')$$

is a viscosity subsolution of (4.1) and the lower limit

$$\underline{z}(t, p, y) := \liminf_{\substack{\epsilon \rightarrow 0^+ \\ (t', p', y') \rightarrow (t, p, y)}} z^\epsilon(t', p', y')$$

is a viscosity supersolution of (4.1).

As \bar{z} and \underline{z} agree at time $t = T$ and satisfy natural growth estimates for large values of p (see Lemma 4.2), we will be able to conclude

$$\bar{z} \leq \underline{z}.$$

Combined with the definitions above, we will also have $\bar{z} = \underline{z} =: \psi$ and that $z^\epsilon \rightarrow \psi$ locally uniformly as $\epsilon \rightarrow 0$ (Remark 6.2 in [6]). First, let us make a basic observation.

Lemma 4.1. \bar{z} and \underline{z} are independent of y .

Proof. 1. As $|z_{y_i}^\epsilon| \leq \sqrt{\epsilon} p_i$ for $i = 1, \dots, n$ in the sense of viscosity solutions,

$$|z^\epsilon(t, p, y_1) - z^\epsilon(t, p, y_2)| \leq \sum_{i=1}^n \sqrt{\epsilon} p_i |(y_1 - y_2) \cdot e_i|, \quad (t, p) \in (0, T) \times (0, \infty)^n, \quad y_1, y_2 \in \mathbb{R}^n. \quad (4.2)$$

Therefore, for $y_1, y_2 \in \mathbb{R}^n$

$$\begin{aligned}
\bar{z}(t, p, y_1) - \bar{z}(t, p, y_2) &= \limsup_{\substack{\epsilon \rightarrow 0^+ \\ (t', p', y'_1) \rightarrow (t, p, y_1)}} z^\epsilon(t', p', y'_1) - \limsup_{\substack{\epsilon \rightarrow 0^+ \\ (t', p', y'_2) \rightarrow (t, p, y_2)}} z^\epsilon(t', p', y'_2) \\
&\leq \limsup_{\substack{\epsilon \rightarrow 0^+ \\ (t', p', y'_1, y'_2) \rightarrow (t, p, y_1, y_2)}} \{z^\epsilon(t', p', y'_1) - z^\epsilon(t', p', y'_2)\} \\
&\leq \limsup_{\substack{\epsilon \rightarrow 0^+ \\ (t', p', y'_1, y'_2) \rightarrow (t, p, y_1, y_2)}} \left\{ \sum_{i=1}^n \sqrt{\epsilon} p_i |(y'_1 - y'_2) \cdot e_i| \right\} \\
&= 0.
\end{aligned}$$

Hence, \bar{z} is independent of y .

2. As

$$(-\underline{z})(t, p, y) := \limsup_{\substack{\epsilon \rightarrow 0^+ \\ (t', p', y') \rightarrow (t, p, y)}} (-z^\epsilon)(t', p', y')$$

and $-z^\epsilon$ also satisfies (4.2), we conclude that \underline{z} is independent of y by the same argument given in part 1. \square

We are finally in position to prove Theorem 1.3. The technique we will use, known as the *perturbed test function method*, is due to L.C. Evans [10, 11] and was first applied to this framework by G. Barles and H. Soner [2]. One difference with the option pricing problem in several assets is that we must work with nonsmooth “correctors” i.e. viscosity solutions u of equation (1.6). We will employ a smoothing argument to overcome this difficulty.

Proof. (of Theorem 1.3) 1. We now proceed to show that \bar{z} is subsolution of (4.1) and \underline{z} is supersolution of (4.1) (with $r = 0$). First assume that $\bar{z} - \phi$ has a local maximum at some point $(t_0, p_0) \in (0, T) \times (0, \infty)^n$ and $\phi \in C^\infty$; for definiteness, we suppose that

$$(\bar{z} - \phi)(t, p) \leq (\bar{z} - \phi)(t_0, p_0), \quad (p_0, t_0) \in \overline{B_\tau}(t_0, p_0).^6$$

We must show

$$-\phi_t(t_0, p_0) - \lambda(d(p_0)D^2\phi(t_0, p_0)d(p_0)) \leq 0. \quad (4.3)$$

By adding $(t, p) \mapsto \eta(|t - t_0|^2 + |p - p_0|^2)$ to ϕ and later sending $\eta \rightarrow 0^+$, we may assume that (t_0, p_0) is a *strict* local maximum point for $\bar{z} - \phi$ in $\overline{B_\tau}(t_0, p_0)$ and also that

$$\det D^2\phi(t_0, p_0) \neq 0.$$

We fix $\delta > 0$ and set

$$\begin{cases} A^\delta(t, p) := (1 + \delta)^2 d(p) D^2\phi(t, p) d(p) \\ A_0 := A^\delta(t_0, p_0) \\ x^{\epsilon, \delta}(t, p, y) := (1 + \delta) d(p) \frac{D\phi(t, p) - y}{\sqrt{\epsilon}} \\ \phi^{\epsilon, \delta}(t, p, y) := \phi(t, p) + \epsilon u(x^{\epsilon, \delta}(t, p, y); A_0) \end{cases}$$

⁶Here $B_r(s, z) := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : |t - s|^2 + |x - z|^2 < r^2\}$.

for $(t, p, y) \in (0, T) \times (0, \infty)^n \times \mathbb{R}^n$. We are assuming that u is a convex, $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$ solution of (1.6) with eigenvalue $\lambda(A_0)$ that satisfies (1.7) and that the set $\Omega = \{x \in \mathbb{R}^n : \max_{1 \leq i \leq n} |u_{x_i}(x)| < 1\}$ is bounded. We established the existence of such a function u in the proof of Theorem 1.1.

2. We claim there is a sequence of positive numbers ϵ_k tending to zero and local maximizers $(t_k, p_k, y_k) \in \overline{B_\tau}(t_0, p_0) \times \mathbb{R}^n$ of $z^{\epsilon_k} - \phi^{\epsilon_k, \delta}$ such that

$$(t_k, p_k) \rightarrow (t_0, p_0),$$

as $k \rightarrow \infty$. We will use the idea presented in appendix of [3] to prove this.

Let $y_0 \in \mathbb{R}^n$ be given and select a sequence $\epsilon_k \rightarrow 0$ and $(t'_k, p'_k, y'_k) \rightarrow (t_0, p_0, y_0)$ as $k \rightarrow \infty$ such that

$$(z^{\epsilon_k} - \phi^{\epsilon_k, \delta})(t'_k, p'_k, y'_k) \rightarrow (\bar{z} - \phi)(t_0, p_0)$$

(recall \bar{z} is independent of the y variable). By estimate (4.10) or (4.11),

$$\limsup_{|y| \rightarrow \infty} \frac{(z^\epsilon - \phi^{\epsilon, \delta})}{\sum_{i=1}^n \sqrt{\epsilon} p_i |y_i|} \leq -\delta < 0$$

locally uniformly in $(t, p) \in (0, T) \times (0, \infty)^n$. Thus, $z^{\epsilon_k} - \phi^{\epsilon_k, \delta}$ has a local maximum at some

$$(t_k, p_k, y_k) \in \overline{B_\tau}(t_0, p_0) \times \mathbb{R}^n$$

for all k sufficiently large.

Recall z^ϵ is a subsolution of the eikonal equation

$$\max_{1 \leq i \leq n} \{|z_{y_i}| - \sqrt{\epsilon} p_i\} = 0.$$

and also that $u \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$. Thus, if $u_{x_i}(x^{\epsilon_k, \delta}) \neq 0$

$$\begin{aligned} |u_{x_i}(x^{\epsilon_k, \delta})| &< |(1 + \delta)u_{x_i}(x^{\epsilon_k, \delta})| \\ &= \frac{1}{\sqrt{\epsilon_k}(p_k \cdot e_i)} |\sqrt{\epsilon_k}(p_k \cdot e_i)(1 + \delta)u_{x_i}(x^{\epsilon_k, \delta})| \\ &= \frac{1}{\sqrt{\epsilon_k}(p_k \cdot e_i)} |D_y \phi^{\epsilon_k, \delta}| \\ &\leq 1. \end{aligned}$$

In either case, $|u_{x_i}(x^{\epsilon_k, \delta})| < 1$ for $i = 1, \dots, n$ and hence

$$|x^{\epsilon_k, \delta}| = (1 + \delta) \left| d(p_k) \frac{D\psi(t_k, p_k) - y_k}{\sqrt{\epsilon_k}} \right| \leq C,$$

for all $k \geq 1$. Therefore, the sequence y_k is bounded.

Without loss of generality, we assume that $(t_k, p_k, y_k) \rightarrow (t_1, p_1, y_1)$, as $k \rightarrow \infty$. Notice that

$$\begin{aligned} (\bar{z} - \phi)(t_1, p_1) &\geq \limsup_{k \rightarrow \infty} (z^{\epsilon_k} - \phi^{\epsilon_k, \delta})(t_k, p_k, y_k) \\ &\geq \limsup_{k \rightarrow \infty} (z^{\epsilon_k} - \phi^{\epsilon_k, \delta})(t'_k, p'_k, y'_k) \\ &= (\bar{z} - \phi)(t_0, p_0). \end{aligned}$$

As $(t_1, p_1) \in \overline{B_\tau}(t_0, p_0)$, it must be that $(t_1, p_1) = (t_0, p_0)$.

3. Since

$$\max_{1 \leq i \leq n} |u_{x_i}(x^{\epsilon_k, \delta})| < 1$$

and u is smooth on the open set Ω ,

$$\lambda(A_0) - \frac{1}{2} \text{tr} \sigma \sigma^t [A_0 + A_0 D^2 u(x^{\epsilon_k, \delta}) A_0 + (x^{\epsilon_k, \delta} + A_0 D u(x^{\epsilon_k, \delta})) \otimes (x^{\epsilon_k, \delta} + A_0 D u(x^{\epsilon_k, \delta}))] = 0,$$

for each $k \geq 1$. Computing as we did in subsection 1.2 we arrive at

$$\begin{aligned} 0 &\geq -\phi_t^{\epsilon_k, \delta} - \frac{1}{2} \text{tr} \left(d(p_k) \sigma \sigma^t d(p_k) \left(D_p^2 \phi^{\epsilon_k, \delta} + \frac{1}{\epsilon_k} (D_p \phi^{\epsilon_k, \delta} - y_k) \otimes (D_p \phi^{\epsilon_k, \delta} - y_k) \right) \right) \\ &= -\phi_t(t_k, p_k) + o(1) \\ &\quad - \frac{1}{2(1+\delta)^2} \text{tr} \sigma \sigma^t [A_0 + A_0 D^2 u(x^{\epsilon_k, \delta}) A_0 + (x^{\epsilon_k, \delta} + A_0 D u(x^{\epsilon_k, \delta})) \otimes (x^{\epsilon_k, \delta} + A_0 D u(x^{\epsilon_k, \delta}))] \\ &= -\phi_t(t_0, p_0) - \frac{1}{(1+\delta)^2} \lambda(A_0) + o(1) \\ &= -\phi_t(t_0, p_0) - \frac{1}{(1+\delta)^2} \lambda((1+\delta)^2 d(p_0) D^2 \phi(t_0, p_0) d(p_0)) + o(1) \end{aligned}$$

as $k \rightarrow \infty$. Therefore, we let $k \rightarrow \infty$ and then $\delta \rightarrow 0^+$ to achieve (4.3).

4. Now assume that $\underline{z} - \phi$ has a local minimum at some point $(t_0, p_0) \in (0, T) \times (0, \infty)^n$ and $\phi \in C^\infty$; for definiteness, we suppose that

$$(\underline{z} - \phi)(t, p) \geq (\underline{z} - \phi)(t_0, p_0), \quad (p_0, t_0) \in \overline{B_\tau}(t_0, p_0).$$

We must show

$$-\phi_t(t_0, p_0) - \lambda(d(p_0) D^2 \phi(t_0, p_0) d(p_0)) \geq 0. \quad (4.4)$$

By subtracting $(t, p) \mapsto \eta(|t - t_0|^2 + |p - p_0|^2)$ from ϕ and later sending $\eta \rightarrow 0^+$, we may assume that (t_0, p_0) is a *strict* local minimum point for $\underline{z} - \phi$ in $\overline{B_\tau}(t_0, p_0)$ and also that

$$\det D^2 \phi(t_0, p_0) \neq 0. \quad (4.5)$$

We fix $\delta \in (0, 1)$, and set

$$\begin{cases} A^\delta(t, p) := (1 - \delta)^2 d(p) D^2 \phi(t, p) d(p) \\ A_0 := A^\delta(t_0, p_0) \\ x^{\epsilon, \delta}(t, p, y) := (1 - \delta) d(p) \frac{D\phi(t, p) - y}{\sqrt{\epsilon}} \\ \phi^{\epsilon, \delta, \rho}(t, p, y) := \phi(t, p) + \epsilon u^\rho(x^{\epsilon, \delta}(t, p, y)) \end{cases}$$

for $(t, p, y) \in (0, T) \times (0, \infty)^n \times \mathbb{R}^n$. Here $u^\rho := \eta^\rho * u$ is the standard mollification of $u = u(\cdot; A_0)$, where u is a convex solution of (1.6) with eigenvalue $\lambda(A_0)$ that satisfies (1.7) and $u \in C_{\text{loc}}^{1, \alpha}(\mathbb{R}^n)$ for any $0 < \alpha < 1$.

5. We claim there is a sequence of positive numbers $\epsilon_k \rightarrow 0$ and local minimizers $(t_k, p_k, y_k) \in \overline{B_\tau}(t_0, p_0) \times \mathbb{R}^n$ of $z^{\epsilon_k} - \phi^{\epsilon_k, \delta, \rho}$ such that

$$(t_k, p_k) \rightarrow (t_0, p_0),$$

as $k \rightarrow \infty$. We will argue as above.

Let $y_0 \in \mathbb{R}^n$ be given and select a sequence $\epsilon_k \rightarrow 0$ and $(t'_k, p'_k, y'_k) \rightarrow (t_0, p_0, y_0)$ as $k \rightarrow \infty$ such that

$$(z^{\epsilon_k} - \phi^{\epsilon_k, \delta, \rho})(t'_k, p'_k, y'_k) \rightarrow (\underline{z} - \phi)(t_0, p_0)$$

(recall \underline{z} is independent of the y variable). By estimate (4.13),

$$\liminf_{|y| \rightarrow \infty} \frac{(z^\epsilon - \phi^{\epsilon, \delta, \rho})}{\sum_{i=1}^n \sqrt{\epsilon p_i} |y_i|} \geq \delta > 0 \quad (4.6)$$

locally uniformly in $(t, p) \in (0, T) \times (0, \infty)^n$ and all ϵ sufficiently small. Thus, $z^{\epsilon_k} - \phi^{\epsilon_k, \delta, \rho}$ has a minimum at some

$$(t_k, p_k, y_k) \in \overline{B_\tau}(t_0, p_0) \times \mathbb{R}^n$$

for all k sufficiently large. Moreover, it must be that y_k is a bounded sequence for if not then (4.6) implies

$$(z^{\epsilon_k} - \phi^{\epsilon_k, \delta, \rho})(t_k, p_k, y_k) \rightarrow +\infty$$

while

$$(z^{\epsilon_k} - \phi^{\epsilon_k, \delta, \rho})(t_k, p_k, y_k) \leq (z^{\epsilon_k} - \phi^{\epsilon_k, \delta, \rho})(t'_k, p'_k, y'_k)$$

and the right hand side above is bounded from above.

Without loss of generality, we assume that $(t_k, p_k, y_k) \rightarrow (t_1, p_1, y_1)$, as $k \rightarrow \infty$. Notice that

$$\begin{aligned} (\underline{z} - \phi)(t_1, p_1) &\leq \liminf_{k \rightarrow \infty} (z^{\epsilon_k} - \phi^{\epsilon_k, \delta, \rho})(t_k, p_k, y_k) \\ &\leq \liminf_{k \rightarrow \infty} (z^{\epsilon_k} - \phi^{\epsilon_k, \delta, \rho})(t'_k, p'_k, y'_k) \\ &= (\underline{z} - \phi)(t_0, p_0). \end{aligned}$$

As $(t_1, p_1) \in \overline{B_\tau}(t_0, p_0)$, it must be that $(t_1, p_1) = (t_0, p_0)$.

6. We have at the point (t_k, p_k, y_k)

$$\begin{aligned} |\phi_{y_i}^{\epsilon_k, \delta}| &= |(1 - \delta)\sqrt{\epsilon_k}(p_k \cdot e_i)u_{x_i}^\rho(x^{\epsilon_k, \delta})| \\ &\leq (1 - \delta)\sqrt{\epsilon_k}(p_k \cdot e_i) \\ &< \sqrt{\epsilon_k}(p_k \cdot e_i) \end{aligned}$$

for $i = 1, \dots, n$. Since z^{ϵ_k} is a viscosity solution of (1.2) and $\phi^{\epsilon_k, \delta, \rho} \in C^2((0, T) \times (0, \infty)^n \times \mathbb{R}^n)$, we compute as in subsection 1.2 to get

$$\begin{aligned} 0 &\leq -\phi_t^{\epsilon_k, \delta, \rho} - \frac{1}{2} \text{tr} \left(d(p_k) \sigma \sigma^t d(p_k) \left(D_p^2 \phi^{\epsilon_k, \delta, \rho} + \frac{1}{\epsilon_k} (D_p \phi^{\epsilon_k, \delta, \rho} - y_k) \otimes (D_p \phi^{\epsilon_k, \delta, \rho} - y_k) \right) \right) \\ &\leq -\phi_t(t_k, p_k) + o(1) \\ &\quad - \frac{1}{2(1 - \delta)^2} \text{tr} \sigma \sigma^t (A_0 + A_0 D^2 u^\rho(x^{\epsilon_k, \delta}) A_0 + (x^{\epsilon_k, \delta} + A_0 D u^\rho(x^{\epsilon_k, \delta})) \otimes (x^{\epsilon_k, \delta} + A_0 D u^\rho(x^{\epsilon_k, \delta}))). \end{aligned} \tag{4.7}$$

As in the proof of Proposition 3.11, we have that

$$\lambda(A_0) - \frac{1}{2} \text{tr} \sigma \sigma^t (A_0 + A_0 D^2 u^\rho A_0 + (x + A_0 D u^\rho) \otimes (x + A_0 D u^\rho)) \leq O(\rho^\alpha) \tag{4.8}$$

for x belonging to a bound subset of \mathbb{R}^n (for some fixed $0 < \alpha < 1$). We also have that as u^ρ is convex and Du^ρ is uniformly bounded that

$$\begin{aligned} \lambda(A_0) - \frac{1}{2} \text{tr} \sigma \sigma^t (A_0 + A_0 D^2 u^\rho A_0 + (x + A_0 D u^\rho) \otimes (x + A_0 D u^\rho)) &\leq \lambda(A_0) - \frac{1}{2} \text{tr} \sigma \sigma^t A_0 \\ &\quad - (|\sigma^t x| - \sqrt{n} |\sigma^t A|)^2 \\ &\leq 0 \end{aligned}$$

for all x large enough. Consequently, (4.8) holds for all $x \in \mathbb{R}^n$.

In particular, (4.8) and (4.7) together imply

$$\begin{aligned} 0 &\leq -\phi_t(t_k, p_k) - \frac{1}{(1 - \delta)^2} \lambda(A_0) + o(1) + O(\rho^\alpha) \\ &= -\phi_t(t_0, p_0) - \frac{1}{(1 - \delta)^2} \lambda(A_0) + o(1) + O(\rho^\alpha) \\ &= -\phi_t(t_0, p_0) - \frac{1}{(1 - \delta)^2} \lambda((1 - \delta)^2 d(p_0) D^2 \phi(t_0, p_0) d(p_0)) + o(1) + O(\rho^\alpha). \end{aligned}$$

We obtain (4.4) by letting $k \rightarrow \infty$ and then $\delta, \rho \rightarrow 0^+$.

7. In order to conclude the proof, we need to argue that $\bar{z} \leq \underline{z}$. Direct computation shows that the function

$$z^\eta(t, p) := \bar{z}(t, p) - \eta \left(\frac{1}{t} + \sum_{i=1}^n p_i \right)$$

is a subsolution of (4.1) (with $r = 0$) for each $\eta > 0$. By Lemma 4.2 below, we have if g satisfies (1.8) then

$$\varphi \leq \underline{z} \leq \bar{z} \leq L,$$

and if g satisfies (1.9) then

$$\varphi \leq \underline{z} \leq \bar{z} \leq L \sum_{i=1}^n p_i. \quad (4.9)$$

Here φ is the Black-Scholes price (that satisfies the PDE (4.12)) and is given by

$$\varphi(p) = \int_{\mathbb{R}^n} g \left(p_1 e^{\sqrt{t}\sigma^t e_1 \cdot z - \frac{1}{2}|\sigma^t e_1|^2 t}, \dots, p_n e^{\sqrt{t}\sigma^t e_n \cdot z - \frac{1}{2}|\sigma^t e_n|^2 t} \right) \frac{e^{-|z|^2/2}}{(2\pi)^{n/2}} dz.$$

When g satisfies (1.9), the explicit formula above with inequality (4.9) gives

$$L = \lim_{|p| \rightarrow \infty} \frac{\varphi(p)}{\sum_{i=1}^n p_i} \leq \lim_{|p| \rightarrow \infty} \frac{\underline{z}(p)}{\sum_{i=1}^n p_i} \leq \lim_{|p| \rightarrow \infty} \frac{\bar{z}(p)}{\sum_{i=1}^n p_i} \leq L.$$

It is now straightforward to check that when g satisfies either (1.8) or (1.9), $z^\eta - \underline{z}$ has a maximum at some $(t_0, p_0) \in (0, T] \times [0, \infty)^n$.

If $t_0 = T$, then

$$\bar{z} \leq \underline{z} + \eta \left(\frac{1}{t} + \sum_{i=1}^n p_i \right).$$

Letting $\eta \rightarrow 0^+$ leads to the desired inequality, $\bar{z} \leq \underline{z}$. Now suppose $t_0 < T$ and, for now, that \bar{z}, \underline{z} are smooth. From calculus,

$$\begin{cases} z_t^\eta(t_0, p_0) = \underline{z}_t(t_0, p_0) \\ d(p_0) D^2 z^\eta(t_0, p_0) d(p_0) \leq d(p_0) D^2 \underline{z}(t_0, p_0) d(p_0) \end{cases}.$$

However, these inequalities would imply a contradiction as

$$\begin{aligned} \frac{\eta}{t_0^2} &= -\bar{z}_t(t_0, p_0) + \underline{z}_t(t_0, p_0) \\ &\leq \lambda(d(p_0) D^2 \bar{z}(t_0, p_0) d(p_0)) - \lambda(d(p_0) D^2 \underline{z}(t_0, p_0) d(p_0)) \\ &= \lambda(d(p_0) D^2 z^\eta(t_0, p_0) d(p_0)) - \lambda(d(p_0) D^2 \underline{z}(t_0, p_0) d(p_0)) \\ &\leq 0. \end{aligned}$$

The last inequality above is due to the monotonicity of λ . It is now routine to use the ideas in Section 8 of [6] to make the same conclusion without assuming smoothness of \bar{z}, \underline{z} . \square

Lemma 4.2. *Let z^ϵ be the solution of (1.2) described in Proposition 1.4.*

(i) *There is a universal constant C such that*

$$\varphi(t, p) \leq z^\epsilon(t, p, y) \leq C + \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i - C|, \quad (t, p, y) \in (0, T] \times (0, \infty)^n \times \mathbb{R}^n, \quad (4.10)$$

provided g satisfies (1.8) or

$$\varphi(t, p) \leq z^\epsilon(t, p, y) \leq C \sum_{i=1}^n p_i + \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i - C|, \quad (t, p, y) \in (0, T] \times (0, \infty)^n \times \mathbb{R}^n, \quad (4.11)$$

provided g satisfies (1.9). Here φ is the “Black-Scholes” price

$$\begin{cases} \varphi_t + \frac{1}{2} \text{tr} \sigma \sigma^t (d(p) D^2 \varphi d(p)) = 0, & (t, p) \in (0, T) \times (0, \infty)^n \\ \varphi = g, & (t, p) \in \{T\} \times (0, \infty)^n \end{cases}. \quad (4.12)$$

(ii) For each $0 < \eta < T$, there is a $K = K(\eta)$ such that

$$z^\epsilon(t, p, y) \geq \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i| - KT\epsilon, \quad (t, p, y) \in (0, T - \eta] \times (0, \infty)^n \times \mathbb{R}^n \quad (4.13)$$

for all $0 < \epsilon < 1/4$.

Proof. (i) We prove (4.11) as the proof for (4.10) is similar. Let $C := L$ be the constant in (1.9) and set $y_0 := (C, C, \dots, C) \in \mathbb{R}^n$. In view of inequality (1.9),

$$\begin{aligned} v^\epsilon(t, p, y_0)|_{t=T} &= 1 - \exp(-(x + y \cdot p - g(p))/\epsilon) \\ &= 1 - \exp\left(-\left(x + C \sum_{i=1}^n p_i - g(p)\right)/\epsilon\right) \\ &\geq 1 - \exp(-x/\epsilon) \end{aligned}$$

As $(t, p) \rightarrow v^\epsilon(t, p, y_0)$ is a supersolution of the backwards parabolic PDE

$$-\psi_t - \frac{1}{2} \text{tr} \sigma \sigma^t (d(p) D^2 \psi d(p)) = 0$$

(recall equation (1.10)), it must be that $v^\epsilon(t, p, y_0) = 1 - \exp(-(x + y_0 \cdot p - z^\epsilon)/\epsilon) \geq 1 - \exp(-x/\epsilon)$ for $(t, p) \in (0, T] \times (0, \infty)^n$. Hence, $z^\epsilon(t, p, y_0) \leq y_0 \cdot p = C \sum_{i=1}^n p_i$ and consequently,

$$\begin{aligned} z^\epsilon(t, p, y) &\leq z^\epsilon(t, p, y_0) + \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i - y_0 \cdot e_i| \\ &\leq C \sum_{i=1}^n p_i + \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i - C| \end{aligned}$$

as claimed.

Similarly, we observe that $(t, x, p, y) \mapsto U_\epsilon(x + y \cdot p - \varphi)$ is a supersolution of (1.10) that satisfies $v^\epsilon|_{t=T} = U_\epsilon(x + y \cdot p - g)$. It follows that $v^\epsilon \leq U_\epsilon(x + y \cdot p - \varphi)$ and thus $z^\epsilon \geq \varphi$.

(ii) We follow the proof of Lemma 2.2 in [2] closely. Let $\eta \in (0, T]$ and $g \in C^\infty(\mathbb{R})$ be a nondecreasing function such that $g(s) = 0$ for $s \leq 0$ and $g(s) = 1$ for $s \geq \eta$. Also set

$$\psi(t, p, y) := g(T - t) \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i| - K\epsilon(T - t)$$

for a positive constant K to be determined below. To establish the inequality (4.11), it suffices to choose K so that ψ is a subsolution of (1.2).

As $0 \leq g \leq 1$, we have $|\psi_{y_i}| \leq \sqrt{\epsilon} p_i$ for $i = 1, \dots, n$. Moreover,

$$\begin{aligned} Q &:= -\psi_t - \frac{1}{2} \text{tr} \left(d(p) \sigma \sigma^t d(p) \left(D_p^2 \psi + \frac{1}{\epsilon} (D_p \psi - y) \otimes (D_p \psi - y) \right) \right) \\ &= -\epsilon K + g' \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i| - \frac{1}{2\epsilon} |\sigma^t d(p) (D_p \psi - y)|^2 \\ &\leq -\epsilon K + g' \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i| - \frac{1}{2\epsilon |\sigma^{-1}|^2} \sum_{i=1}^n p_i^2 (g \sqrt{\epsilon} |y_i| - y_i)^2 \\ &\leq -\epsilon K + g' \sum_{i=1}^n \sqrt{\epsilon} p_i |y_i| - \frac{1}{2\epsilon |\sigma^{-1}|^2} \sum_{i=1}^n p_i^2 y_i^2 (1 - \sqrt{\epsilon})^2 \\ &\leq \epsilon \left(-K + \frac{\epsilon}{2} \left(\frac{|\sigma^{-1}| g' \sqrt{\epsilon}}{1 - \sqrt{\epsilon}} \right)^2 \right) \\ &\leq 0 \end{aligned}$$

for $K = K(\eta)$ chosen large enough. K can be chosen independent of ϵ as $\epsilon \sqrt{\epsilon} / (1 - \sqrt{\epsilon}) < 1/4$ for $\epsilon \in (0, 1/4)$. \square

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A Monotonicity of $A \mapsto \text{tr} f(A)$

In this appendix, we prove Proposition 3.7, which we restate for the reader's convenience.

Proposition A.1. *Let $f \in C(\mathbb{R})$ be monotone non-decreasing. Then $\mathcal{S}(n) \ni A \mapsto \text{tr} f(A)$ is monotone non-decreasing with respect to the partial ordering on $\mathcal{S}(n)$.*

We first prove a version of this proposition for smooth functions.

Proposition A.2. *Let $f \in C^\infty(\mathbb{R})$ and $N \in \mathcal{S}(n)$. For each $A \in \mathcal{S}(n)$, we have*

$$D \text{tr}[f(A)]N := \lim_{t \rightarrow 0} \frac{\text{tr} f(A + tN) - \text{tr} f(A)}{t} = \text{tr}[f'(A)N]. \quad (\text{A.1})$$

Proof. $A = O\Lambda O^t$, where $O^t O = I_n$ and Λ is diagonal; we shall also write $A = (a_{ij})_{1 \leq i, j \leq n}$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. With this notation,

$$\text{tr} f(A) = \sum_{i=1}^n f(\lambda_i).$$

Now $\lambda_k = A O e_k \cdot O e_k$ and $A = \sum_{i,j=1}^n a_{ij} e_i e_j^t$. Therefore

$$\begin{aligned} \frac{\partial \lambda_k}{\partial a_{ij}} &= e_i e_j^t O e_k \cdot O e_k + A \frac{\partial}{\partial a_{ij}} O e_k \cdot O e_k + A O e_k \cdot \frac{\partial}{\partial a_{ij}} O e_k \\ &= e_j^t O e_k \cdot e_i^t O e_k + 2 A O e_k \cdot \frac{\partial}{\partial a_{ij}} O e_k \\ &= O_{ik} O_{jk} + 2 \lambda_k O e_k \cdot \frac{\partial}{\partial a_{ij}} O e_k \\ &= O_{ik} O_{jk} + \lambda_k \frac{\partial}{\partial a_{ij}} |O e_k|^2 \\ &= O_{ik} O_{jk}, \end{aligned}$$

since $O^t O = I_n$.

Let $N \in \mathcal{S}(n)$. We have from the definition of $D\text{tr}[f(A)]N$ in (A.1), the chain rule, and the above computations that

$$\begin{aligned} D\text{tr}[f(A)]N &= \sum_{i,j=1}^n \frac{\partial}{\partial a_{ij}} \text{tr} f(A) N_{ij} \\ &= \sum_{i,j=1}^n \sum_{k=1}^n f'(\lambda_k) \frac{\partial \lambda_k}{\partial a_{ij}} N_{ij} \\ &= \sum_{k=1}^n f'(\lambda_k) \sum_{i,j=1}^n O_{ik} O_{jk} N_{ij} \\ &= \sum_{k=1}^n f'(\lambda_k) (O^t N O)_{kk} \\ &= \text{tr}[f'(A)N]. \end{aligned}$$

□

Lemma A.3. *Let $f \in C^\infty(\mathbb{R})$ be non-decreasing. Then $\mathcal{S}(n) \ni A \mapsto \text{tr} f(A)$ is non-decreasing.*

Proof. Let $A, B \in \mathcal{S}(n)$ with $B \geq A$. From the above proposition,

$$\begin{aligned} \operatorname{tr} f(B) - \operatorname{tr} f(A) &= \int_0^1 \frac{d}{dt} \operatorname{tr} f(A + t(B - A)) dt \\ &= \int_0^1 D \operatorname{tr} f(A + t(B - A))(B - A) dt \\ &= \int_0^1 \operatorname{tr}[f'(A + t(B - A))(B - A)] dt \\ &\geq 0, \end{aligned}$$

as the matrix $f'(A + t(B - A)) \geq 0$ for all $t \in [0, 1]$ and $B - A \geq 0$. \square

Proof. (Proposition (3.7)) Let f^ϵ denote the standard mollifier of f and suppose $A, B \in \mathcal{S}(n)$ with $B \geq A$. By the above lemma, $\operatorname{tr} f^\epsilon(B) \geq \operatorname{tr} f^\epsilon(A)$ for all $\epsilon > 0$ as f^ϵ is non-decreasing. Letting $\epsilon \rightarrow 0^+$ implies $\operatorname{tr} f(B) \geq \operatorname{tr} f(A)$. \square

B An elementary matrix inequality

Lemma B.1. *Let $a, b, c \in \mathcal{S}(n)$ and b, c nonnegative definite. Further assume that*

$$a(\xi_1 + \xi_2) \cdot (\xi_1 + \xi_2) \leq b\xi_1 \cdot \xi_1 + c\xi_2 \cdot \xi_2. \quad (\text{B.1})$$

for all $\xi_1, \xi_2 \in \mathbb{R}^n$. Then for any $A_1, A_2 \in \mathcal{S}(n)$,

$$\operatorname{tr}[(A_1 + A_2)^2 a] \leq \operatorname{tr}[bA_1^2] + \operatorname{tr}[cA_2^2].$$

Proof. 1. First assume that b and c are positive definite. Then for a given $\xi_1 \in \mathbb{R}^n$, we may select ξ_2 such that

$$b\xi_1 = c\xi_2.$$

Also note that for these vectors

$$\begin{cases} (b + c)\xi_1 = c(\xi_1 + \xi_2) \\ (b + c)\xi_1 = b(\xi_1 + \xi_2) \end{cases}.$$

Therefore,

$$\begin{aligned} b\xi_1 \cdot \xi_1 &= b(b + c)^{-1}c(\xi_1 + \xi_2) \cdot (b + c)^{-1}c(\xi_1 + \xi_2) \\ &= c(b + c)^{-1}b(b + c)^{-1}c(\xi_1 + \xi_2) \cdot (\xi_1 + \xi_2) \\ &= c^2b(b + c)^{-2}(\xi_1 + \xi_2) \cdot (\xi_1 + \xi_2), \end{aligned}$$

where we are using the fact that all the matrices being manipulated are *symmetric*.

Similarly

$$c\xi_2 \cdot \xi_2 = b^2c(b + c)^{-2}(\xi_1 + \xi_2) \cdot (\xi_1 + \xi_2),$$

and so

$$\begin{aligned}
b\xi_1 \cdot \xi_1 + c\xi_2 \cdot \xi_2 &= (b^2c(b+c)^{-2} + c^2b(b+c)^{-2}) (\xi_1 + \xi_2) \cdot (\xi_1 + \xi_2) \\
&= (c^2b + b^2c)(b+c)^{-2}(\xi_1 + \xi_2) \cdot (\xi_1 + \xi_2) \\
&= bc(b+c)^{-1}(\xi_1 + \xi_2) \cdot (\xi_1 + \xi_2).
\end{aligned}$$

As the vector sum $\xi_1 + \xi_2$, where $b\xi_1 = c\xi_2$, runs through all vectors in \mathbb{R}^n , we have by assumption (B.1)

$$a \leq bc(b+c)^{-1}.$$

Also observe that symmetric matrices obey the Cauchy–Schwarz inequality:

$$d_1d_2 \leq \frac{d_1^2 + d_2^2}{2}, \quad d_1, d_2 \in \mathcal{S}(n).$$

With these observations, we have

$$\begin{aligned}
\text{tr} [(A_1 + A_2)^2 a] &= \text{tr}[A_1^2 a + 2A_1 A_2 a + A_2^2 a] \\
&= \text{tr}[A_1^2 a + 2\sqrt{b^{-1}c}A_1 \sqrt{c^{-1}b}A_2 a + A_2^2 a] \\
&\leq \text{tr}[(I_n + b^{-1}c)A_1^2 + (I_n + c^{-1}b)A_2^2]a \\
&\leq \text{tr}[(I_n + b^{-1}c)A_1^2 + (I_n + c^{-1}b)A_2^2]bc(b+c)^{-1} \\
&= \text{tr}[bc(b+c)^{-1}(I_n + b^{-1}c)A_1^2] + \text{tr}[bc(b+c)^{-1}(I_n + c^{-1}b)A_2^2] \\
&= \text{tr}[bA_1^2] + \text{tr}[cA_2^2].
\end{aligned}$$

2. If b, c are only nonnegative definite, we replace them by $b + \delta I_n$ and $c + \delta I_n$ for $\delta > 0$ and small. We can then repeat the computation above and pass to the limit as $\delta \rightarrow 0^+$ to get the desired inequality. \square

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